Maxwell's Equations

$$\begin{cases} \nabla \cdot \mathbf{E} = \rho / \varepsilon_0 & \text{Gauss' law} \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} & \text{Faraday's law} \\ \nabla \cdot \mathbf{B} = 0 & \text{No name} \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} & \text{Ampere's law} \end{cases}$$

Electro- and Magnetostatics:

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$$

$$\nabla \times \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

ρ and **J** are the sources of the electric and magnetic fields. Static magnetic fields have no source: Magnetic monopoles as 'magnetic charges' have never been observed in nature, although the whole theory of electromagnetism could be described including such magnetic charges.

Simple Examples:

What is the electric field generated by a point charge?

Need Gauss Law in integral form!

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$$

$$\oint_V \nabla \cdot \mathbf{E} d\tau = \frac{1}{\varepsilon_0} \oint_V \rho d\tau$$

$$\oint_S \mathbf{E} d\mathbf{a} = \frac{q}{\varepsilon_0}$$

$$E \times 4\pi r^2 = \frac{q}{\varepsilon_0}$$

$$\mathbf{E} = E\hat{\mathbf{r}} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}$$

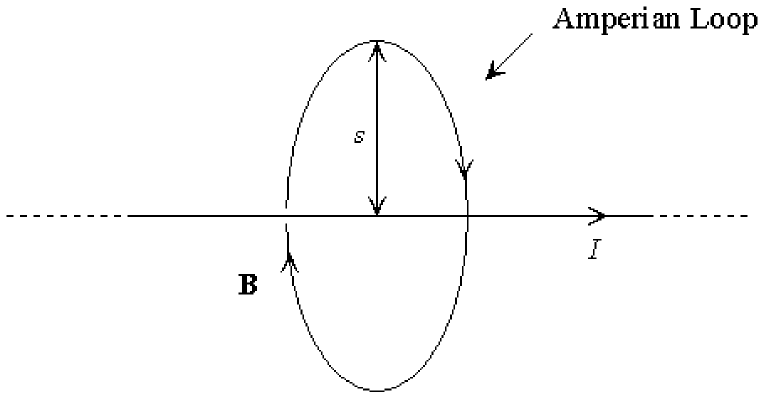
What is the magnetic field generated by a linear wire carrying a continuous current I?

Need Ampere's law in integral form

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

$$\oint_S \nabla \times \mathbf{B} d\mathbf{a} = \mu_0 \oint_S \mathbf{J} d\mathbf{a}$$

$$\oint_P \mathbf{B} d\mathbf{l} = \mu_0 I_{encl.}$$



$$\oint \mathbf{B} \cdot d\mathbf{l} = B \cdot 2\pi s = \mu_0 I$$

$$\therefore \qquad \mathrm{B} = \frac{\mu_0 I}{2\pi s}$$

Further important laws:

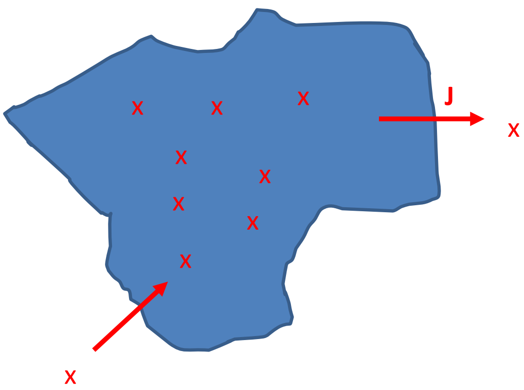
Force law:

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

The force law is especially important as the existence of forces justifies the concept of electric and magnetic fields. The fields themselves cannot be measured directly.

Continuity equation

The continuity equation describes the **local conservation of charges**. Consider an arbitrary volume which contains a certain number of charges. The continuity equation states that if the number of charges within this volume is changing as a function of time, than there must be a current flowing across the boundary.



Consider the total charge Q in a volume V. This charge may vary as a function of time as charge is moving in and out through the boundary of the volume.

$$Q(t) = \oint_V \rho(\mathbf{r},t)d\tau$$

Current through boundary:

$$\frac{dQ}{dt} = -\oint_S \mathbf{J}d\mathbf{a}$$ Using the divergence theorem:

$$\oint_V \frac{\partial\rho}{\partial t}d\tau = -\oint_V \nabla\mathbf{J}d\tau$$

As this must be true for any volume we obtain the **Continuity Equation**:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = 0$$

Maxwell's Equations in vacuum

$$\begin{cases} \nabla\cdot\mathbf{E} = 0 \\ \nabla\times\mathbf{E} = -\dfrac{\partial\mathbf{B}}{\partial t} \\ \nabla\cdot\mathbf{B} = 0 \\ \nabla\times\mathbf{B} = \mu_0\varepsilon_0\dfrac{\partial\mathbf{E}}{\partial t} \end{cases}$$

Electromagnetic Waves in Vacuum

Taking the curl on both sides of the Faraday's law, we have

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t} \nabla \times \mathbf{B}$$

By the Ampere's law,

$$\nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{\partial}{\partial t} \left(\mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

By Gauss' law,

$$\nabla^2 \mathbf{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Similarly, by taking the curl on both sides of the Ampere's law, we have

$$\nabla \times (\nabla \times \mathbf{B}) = \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \nabla \times \mathbf{E}$$

By Faraday's law

$$\nabla (\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = -\mu_0 \varepsilon_0 \frac{\partial}{\partial t} \frac{\partial \mathbf{B}}{\partial t}$$

Since

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla^2 \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

Therefore, both the E field and B field satisfy the wave equation and admit solution of propagating waves.

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$$

→ speed of EM wave

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \frac{1}{\sqrt{4\pi \times 10^{-7} \times 8.85 \times 10^{-12}}} = 3.00 \times 10^8 \, \mathrm{ms}^{-1}$$