Energy

Energy is an extensive property of a physical system, which cannot be measured or observed directly. However it can be calculated from its state and observed indirectly by the capacity of a system to perform work. The SI unit of energy is the joule (J) (equivalent to a newton-meter or a watt-second). One reason for the importance of energy in physics is that it is a conserved quantity. It can neither be created nor destroyed, but it can be changed into different forms. Energy can take many different forms such as thermal energy, kinetic energy, potential energy, electromagnetic energy, mass, dark energy, etc. The sum of all the forms of energy inside a volume of space can only change by the amount of energy leaving or entering the volume.

Not all of the energy in a system can be converted into work, however. The quantity of energy of a system that can be converted to work is called the available energy. Thermal energy has a special status; as the most disordered, highest entropy form of energy, the second law of thermodynamics limits the amount of thermal energy that can be converted into other forms of energy.

Energy is necessary for things to change. All living things require available energy to stay alive; humans get such energy from food, along with the oxygen needed to metabolize the food. Human civilization requires a continual supply of energy to function, energy resources such as fossil fuels are a vital topic in economics and politics. Earth's climate and ecosystem are driven by the radiant energy Earth receives from the sun, and are delicately sensitive to changes in the amount received. (Wikipedia)

Energy in Electromagnetic Fields: The Poynting's Theorem

Work to assemble a static charge distribution:

$$W_e = \frac{\varepsilon_0}{2}\int_{\mathcal{V}} E^2 d\tau$$ (We will derive this in Chapter 2)

Work to drive an electric current:

$$W_e = \frac{1}{2\mu_0}\int_{\mathcal{V}} B^2 d\tau$$ (We will derive this in Chapter 5)

Total energy in a volume *V*:

$$U_{em} = \frac{1}{2}\int_{\mathcal{V}}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)d\tau$$

Work on charges within this volume during a time interval *dt*:

$$\mathbf{F}\cdot d\mathbf{l} = q\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)\cdot\mathbf{v}dt = q\mathbf{E}\cdot\mathbf{v}dt$$

With $q=\rho d\tau$ and $\rho\mathbf{v}=\mathbf{J}$

$$\frac{dW_{mech}}{dt} = \int_{\mathcal{V}} q\mathbf{E}\cdot\mathbf{v} = \int_{\mathcal{V}} \rho\mathbf{v}\cdot\mathbf{E}d\tau = \int_{\mathcal{V}} \mathbf{J}\cdot\mathbf{E}d\tau$$

With $\mathbf{J}\cdot\mathbf{E}$ as the work per unit of time and per unit volume.

Use Ampere Maxwell's Law to eliminate **J**:

$$\begin{aligned}\frac{dW_{\text{mech}}}{dt} &= \int_{\mathcal{V}}\left[\frac{1}{\mu_0}\left(\nabla\times\mathbf{B}\right)-\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\right]\cdot\mathbf{E}d\tau \\ &= \int_{\mathcal{V}}\frac{1}{\mu_0}\left(\nabla\times\mathbf{B}\right)\cdot\mathbf{E}d\tau-\varepsilon_0\int_{\mathcal{V}}\frac{\partial\mathbf{E}}{\partial t}\cdot\mathbf{E}d\tau \\ &= \int_{\mathcal{V}}\frac{1}{\mu_0}\left(\nabla\times\mathbf{B}\right)\cdot\mathbf{E}d\tau-\frac{\varepsilon_0}{2}\int_{\mathcal{V}}\frac{\partial}{\partial t}\left(\mathbf{E}\cdot\mathbf{E}\right)d\tau\end{aligned}$$

Hence

$$\frac{dW_{\text{mech}}}{dt}+\frac{\varepsilon_0}{2}\int_{\mathcal{V}}\frac{\partial}{\partial t}\left(\mathbf{E}\cdot\mathbf{E}\right)d\tau = \int_{\mathcal{V}}\frac{1}{\mu_0}\left(\nabla\times\mathbf{B}\right)\cdot\mathbf{E}d\tau$$

Consider:

$$\nabla\cdot\left(\mathbf{E}\times\mathbf{B}\right)=\mathbf{B}\cdot\left(\nabla\times\mathbf{E}\right)-\mathbf{E}\cdot\left(\nabla\times\mathbf{B}\right)$$

Hence

$$\begin{aligned}\left(\nabla\times\mathbf{B}\right)\cdot\mathbf{E} &= \mathbf{B}\cdot\left(\nabla\times\mathbf{E}\right)-\nabla\cdot\left(\mathbf{E}\times\mathbf{B}\right) \\ &= \mathbf{B}\cdot\left(-\frac{\partial\mathbf{B}}{\partial t}\right)-\nabla\cdot\left(\mathbf{E}\times\mathbf{B}\right) \\ &= -\frac{1}{2}\frac{\partial}{\partial t}\left(\mathbf{B}\cdot\mathbf{B}\right)-\nabla\cdot\left(\mathbf{E}\times\mathbf{B}\right)\end{aligned}$$

$$\begin{aligned}\frac{dW_{\text{mech}}}{dt}+\frac{\varepsilon_0}{2}\int_{\mathcal{V}}\frac{\partial}{\partial t}\left(\mathbf{E}\cdot\mathbf{E}\right)d\tau &= \int_{\mathcal{V}}\frac{1}{\mu_0}\left(\nabla\times\mathbf{B}\right)\cdot\mathbf{E}d\tau \\ &= -\int_{\mathcal{V}}\left[\frac{1}{2\mu_0}\frac{\partial}{\partial t}\left(\mathbf{B}\cdot\mathbf{B}\right)+\frac{1}{\mu_0}\nabla\cdot\left(\mathbf{E}\times\mathbf{B}\right)\right]d\tau\end{aligned}$$

$$\frac{dW_{\text{mech}}}{dt}+\frac{d}{dt}\int_{\mathcal{V}}\frac{1}{2}\varepsilon_0 E^2 d\tau+\frac{d}{dt}\int_{\mathcal{V}}\frac{1}{2\mu_0}B^2 d\tau=-\oint_S\frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a}$$

This is the Work-Energy Theorem of Electrodynamics

We can interpret

$$W_{\text{em}}=\int_{\mathcal{V}}\frac{1}{2}\varepsilon_0 E^2 d\tau+\int_{\mathcal{V}}\frac{1}{2\mu_0}B^2 d\tau$$

As the energy carried by the electromagnetic fields.

Then,

$$\frac{d}{dt}(W_{\text{mech}}+W_{\text{em}})=-\oint_S\frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a}$$

implying that the rate of change of total energy in *V*, which is the sum of the mechanical energies of the particles and the field energies, equals negative the flux of the vector

$$\mathbf{S}=\frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})$$

through the surface *S* enclosing the volume.

Physical Interpretation:

The vector S is called the Poynting vector, which then obviously has the physical meaning of energy flux density.

In other words, $\mathbf{S}\cdot d\mathbf{a}$ is the *energy per unit time* crossing the infinitesimal area $d\mathbf{a}$.

Let u_{mech} and $u_{\text{em}}=\frac{1}{2}\varepsilon_0 E^2+\frac{1}{2\mu_0}B^2$ denote the mechanical energy density and the field energy density, respectively.

$$\begin{cases} W_{\text{mech}}=\int_{\mathcal{V}} u_{\text{mech}}\, d\tau' \\ W_{\text{em}}=\int_{\mathcal{V}} u_{\text{em}}\, d\tau' \end{cases}$$

Then

$$\frac{d}{dt}\left(\int_{\mathcal{V}} u_{\text{mech}}\,d\tau + \int_{\mathcal{V}} u_{\text{em}}\,d\tau\right) = -\int_{S} \mathbf{S}\cdot d\mathbf{a}$$

$$\Rightarrow \int_{\mathcal{V}} \frac{\partial}{\partial t}\left(u_{\text{mech}} + u_{\text{em}}\right) d\tau = -\int_{S} \mathbf{S}\cdot d\mathbf{a}$$

By divergence theorem:

$$\frac{\partial}{\partial t}\left(u_{\text{mech}} + u_{\text{em}}\right) + \nabla\cdot\mathbf{S} = 0$$

Comparing the above equation with the continuity equation of charges:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\mathbf{J} = 0$$

We can conclude that the differential form of Poynting's theorem is in fact the continuity equation of energy and thus represents the law of the conservation of energy.