Potentials and Fields

Scalar and Vector Potentials

How do the sources (ρ and **J**) generate the electric and magnetic fields in the general (dynamic) case?

$$\begin{cases} \nabla \cdot \mathbf{E} = \rho / \varepsilon_0 & \text{Gauss' law} \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} & \text{Faraday's law} \\ \nabla \cdot \mathbf{B} = 0 & \text{No name} \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} & \text{Ampere's law with Maxwell's correction} \end{cases}$$

Given $\rho(\mathbf{r}, t)$ and $\mathbf{J}(\mathbf{r}, t)$ what are the fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$?

Static case: Coulomb and Biot Savart Law.

$$\nabla \times \mathbf{E} = 0$$

allowed us to write **E** as the gradient of a scalar potential:

$$\mathbf{E} = -\nabla V$$

What about the dynamic (time dependent) case?

The curl of E is non-zero, cannot define a simple scalar potential!

However: $\nabla \cdot \mathbf{B} = 0$

remains true, therefore we can still define the vector potential **A**:

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Putting $\mathbf{B} = \nabla \times \mathbf{A}$ into Faraday's Law yields:

$$\nabla \times \mathbf{E} = \frac{\partial}{\partial t} \nabla \times \mathbf{A}$$

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

This is a function whose curl does vanish!

It can thus be written as the gradient of a scalar:

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V$$

$$or:$$

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

If **A** constant, electrostatic case!

Putting $\mathbf{E} = -\nabla V - \dfrac{\partial \mathbf{A}}{\partial t}$ into $\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$ yields:

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\varepsilon_0} \qquad \text{(i)}$$

(Dynamic version of Poisson's equation)

Putting $\mathbf{B} = \nabla \times \mathbf{A}$ into $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \dfrac{\partial \mathbf{E}}{\partial t}$ yields:

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J} - \mu_0 \varepsilon_0 \nabla \frac{\partial V}{\partial t} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}$$

Which can be rearranged as:

$$\left(\nabla^2 \mathbf{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) - \nabla \left(\nabla \cdot \mathbf{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t} \right) = -\mu_0 \mathbf{J} \qquad \text{(ii)}$$

These 2 equations contain all information in the Maxwell's equations, expressed in terms of scalar and vector potentials!

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\varepsilon_0}$$

$$\left(\nabla^2 \mathbf{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) - \nabla\left(\nabla \cdot \mathbf{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 \mathbf{J}$$

These equations are too complicated, but we can simplify them by imposing extra conditions on *V* and **A**, as long as this does not change **E** and **B**!

$\Rightarrow$ **Gauge Freedom / Gauge Transformations**

Suppose we have two sets of potentials (V, **A**) and V', **A'**), which correspond to the same electric and magnetic fields.

Write $\mathbf{A}' = \mathbf{A} + \boldsymbol{\alpha}$ and $V' = V + \beta$

Since the two **A** potentials give the same **B**, their curls must be equal!

Hence $\nabla \times \boldsymbol{\alpha} = 0$ and $\boldsymbol{\alpha}$ can be written as the gradient of a scalar:

$$\boldsymbol{\alpha} = \nabla \lambda$$

As the two potentials also give the same **E** field ($\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$):

$$\nabla \beta + \frac{\partial \boldsymbol{\alpha}}{\partial t} = 0 \quad \text{or} \quad \nabla\left(\beta + \frac{\partial \lambda}{\partial t}\right) = 0$$

The term $\beta + \frac{\partial \lambda}{\partial t}$ does not depend on the position but may depend on time:

$$k(t) = \beta + \frac{\partial \lambda}{\partial t}$$

Let's absorb $k(t)$ into λ, as this will not affect the gradient of λ. It just adds $k(t)$ to $d\lambda/dt$.

Then: $$\mathbf{A}' = \mathbf{A} + \nabla \lambda \qquad V' = V - \frac{\partial \lambda}{\partial t}$$

For any scalar function λ, we can add $\nabla \lambda$ to **A**, provided we simultaneously subtract $\frac{\partial \lambda}{\partial t}$ from V. This will not change **E** and **B**.

Such changes are called **gauge transformations**

The Coulomb Gauge

$$\nabla \cdot \mathbf{A} = 0$$

Then: $$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\varepsilon_0}$$

Is simplified to: $$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

No time dependence, source charge creates *V* immediately everywhere in space.
At odds with special relativity!

Solution: $$V(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}',t)}{\mathsf{r}}\, d\tau'$$

However, *V* alone does not tell you **E**, need to know **A** as well!

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

A contains the time dependence!

Although *V* is particularly simple to calculate, **A** is particularly complicated:

$$\left(\nabla^2 \mathbf{A} - \mu_0\varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) = -\mu_0 \mathbf{J} + \mu_0\varepsilon_0 \nabla \frac{\partial V}{\partial t}$$

The Lorentz Gauge

$$\nabla \cdot \mathbf{A} = -\mu_0 \varepsilon_0 \frac{\partial V}{\partial t}$$

$$\nabla^2 \mathbf{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J}$$

$$\nabla^2 V - \mu_0 \varepsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon_0}$$

Both potentials now contain a time dependence!

4 dimensional version of the Poisson Equation!

Can be simplified with the d'Alembertian operator:

$$\nabla^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} = \Box^2$$

$$\Box^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

$$\Box^2 V = -\frac{\rho}{\varepsilon_0}$$