**Momentum and Angular Momentum of the Electromagnetic Fields**

We have found out that the magnetic forces between two charged particles do not necessarily fulfill the third law of Newton that force equals the counterforce. However, any moving charged particle is accompanied by the electric and magnetic field it produces and thus this discrepancy is caused by the fact that we must attribute momentum also to the fields.

To derive an expression for the momentum carried by the fields, we start with the force law. Consider an arbitrary volume $V$ with boundary $S$, filled with a certain charge distribution. The total force on all the charges is

$$\mathbf{F} = \int_V (\mathbf{E} + \mathbf{v} \times \mathbf{B})\rho d\tau = \int_V (\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}) d\tau$$

If we consider instead the force per unit volume we eliminate the volume integral

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}$$

What we want is an expression which depends only on the **E** and **B** fields and not the sources. Therefore we are using Gauss' law and Ampere-Mawell's law to eliminate $\rho$ and **J**.

$$\mathbf{f} = \varepsilon_0(\boldsymbol{\nabla}\mathbf{E})\mathbf{E} + \left(\frac{1}{\mu_0}\nabla \times \mathrm{B} - \varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\right) \times \mathbf{B}$$

With

$$\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) = \left(\frac{\partial\mathbf{E}}{\partial t} \times \mathbf{B}\right) + \left(\mathbf{E} \times \frac{\partial\mathbf{B}}{\partial t}\right)$$

and

$$\frac{\partial\mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

we obtain

$$\frac{\partial\mathbf{E}}{\partial t} \times \mathbf{B} = \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) + \mathbf{E} \times (\nabla \times \mathbf{E})$$

Inserting this back into the expression for the force per unit volume, we obtain

$$\mathbf{f} = \boldsymbol{\epsilon_0}[(\boldsymbol{\nabla}\mathbf{E})\mathbf{E} - \mathbf{E} \times (\boldsymbol{\nabla} \times \mathbf{E})] + \frac{\mathbf{1}}{\boldsymbol{\mu_0}}[(\boldsymbol{\nabla}\mathbf{B})\mathbf{B} - \mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{B})] - \boldsymbol{\epsilon_0}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial t}}(\mathbf{E} \times \mathbf{B})$$

Here we have added the term $(\nabla\mathbf{B})\mathbf{B}$ , which does not change anything as the divergence of **B** is zero anyway, in order to make this expression more symmetric with respect to the **E** and **B** field.

With the product rule

$$\nabla(E^2) = 2(\mathbf{E} \cdot \nabla)\mathbf{E} + 2\mathbf{E} \times (\nabla \times \mathbf{E})$$

$$\mathbf{E} \times (\nabla \times \mathbf{E}) = \frac{1}{2}\nabla(E^2) - (\mathbf{E} \cdot \nabla)\mathbf{E}$$

and the same procedure applies for the **B** field.

$$\nabla(B^2) = 2(\mathbf{B}\cdot\nabla)\mathbf{B} + 2\mathbf{B}\times(\nabla\times\mathbf{B})$$

$$\mathbf{B}\times(\nabla\times\mathbf{B}) = \frac{1}{2}\nabla(B^2) - (\mathbf{B}\cdot\nabla)\mathbf{B}$$

We obtain

$$\mathbf{f} = \epsilon_0[(\boldsymbol{\nabla}\mathbf{E})\mathbf{E} + (\mathbf{E}\cdot\nabla)\mathbf{E}] + \frac{1}{\mu_0}[(\boldsymbol{\nabla}\mathbf{B})\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{B}]$$
$$-\frac{1}{2}\boldsymbol{\nabla}\left(\boldsymbol{\epsilon_0}E^2 + \frac{\mathbf{1}}{\boldsymbol{\mu_0}}B^2\right) - \epsilon_0\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B})$$

This expression looks very complicated, however we can express the first 3 terms as a tensor (Maxwell stress tensor). The last term can be expressed in terms of the Poynting vector $-\epsilon_0\mu_0\frac{\boldsymbol{\partial}\mathbf{S}}{\boldsymbol{\partial t}}$ .

Let's look a bit closer on the **Maxwell stress tensor**.
We define it as

$$T_{\boldsymbol{ij}} = \epsilon_0\left[E_iE_j - \frac{1}{2}\delta_{ij}E^2\right] + \frac{1}{\mu_0}\left[B_iB_j - \frac{1}{2}\delta_{ij}B^2\right]$$

*i* and *j* are the coordinates *x*, *y* and *z* and thus the tensor has 9 components.
The diagonal elements are e.g.:

$$T_{xx} = \frac{1}{2}\epsilon_0\left[E_x^2 - E_y^2 - E_z^2\right] + \frac{1}{2\mu_0}\left[B_x^2 - B_y^2 - B_z^2\right]$$

And the off-diagonal elements e.g.:

$$T_{xy} = \epsilon_0\left[E_x\ E_y\ \right] + \frac{1}{2\mu_0}\left[B_x\ B_y\ \right]$$

The jth component of the divergence of $\overleftrightarrow{\boldsymbol{T}}$ is:

$$\left(\nabla\cdot\overleftrightarrow{\boldsymbol{T}}\right) = \epsilon_0\left[(\boldsymbol{\nabla}\mathbf{E})E_j - (\mathbf{E}\cdot\nabla)E_j - \frac{1}{2}\boldsymbol{\nabla}_jE^2\right] + \frac{1}{\mu_0}\left[(\boldsymbol{\nabla}\mathbf{B})B_j - (\mathbf{B}\cdot\nabla)B_j - \frac{1}{2}\boldsymbol{\nabla}_jB^2\right]$$

With the stress tensor we can write the force per unit volume in the very compact form

$$\mathbf{f} = \boldsymbol{\nabla}\overleftrightarrow{\boldsymbol{T}} - \epsilon_0\mu_0\frac{\partial\mathbf{S}}{\boldsymbol{\partial t}}$$

The total electromagnetic force on all charges in the volume *V* is then

$$\mathbf{F} = \int_V \boldsymbol{\nabla}\overleftrightarrow{\boldsymbol{T}}\,d\tau - \int_V \epsilon_0\,\mu_0\frac{\partial\mathbf{S}}{\partial t}d\tau = \int_S \overleftrightarrow{\boldsymbol{T}}\,\boldsymbol{d}\mathbf{a} - \int_V \epsilon_0\,\mu_0\frac{\partial\mathbf{S}}{\partial t}d\tau$$

In the static case, the second term is evidently zero. The stress tensor is the force or stress per unit area acting on the surface of our volume, generated by the electric and magnetic fields. Diagonal elements represent pressures and off diagonal elements represent shear forces. This tensor represents thus the 'mechanical' properties of this volume due to the presence of electrical fields. This is analogous to the elastic properties to a solid material, which is described by a similar tensor containing the elastic constants of the material along the different directions in the solid.

To understand the significance of the second term, let's calculate the momentum from the force. The force on an object is equal to the rate of change of its momentum. Let $p_{mech}$ be the total mechanical momentum of the charges contained in our volume.

$$\mathbf{F} = \frac{d\boldsymbol{p}_{\boldsymbol{mech}}}{dt}$$

$$\frac{d\boldsymbol{p}_{\boldsymbol{mech}}}{dt} = -\epsilon_0\mu_0\frac{\partial}{\partial t}\int_V \mathbf{S}\,d\tau + \int_S \overleftrightarrow{\boldsymbol{T}}\,d\mathbf{a}$$

The first term is therefore the momentum stored in the electromagnetic fields $\mathbf{p}_{em}$. Any increase of the momentum must flow in through the surface

To derive an expression which is independent of the chosen volume of our charge configuration we can define the density of momentum in the fields.

$$\wp_{em} = \epsilon_0\mu_0\mathbf{S}$$

With this we can obtain the following equation which represents the continuity equation for the momentum and thus the law of conservation of momentum

$$\frac{d}{dt} = \left(\wp_{mech} + \wp_{em}\right) = \boldsymbol{\nabla}\overleftrightarrow{\boldsymbol{T}}$$

$-\overleftrightarrow{\boldsymbol{T}}$ is the momentum flux density, which plays the role of the current density **J.**

From this we can define also an angular momentum density. The Poynting vector **S** is the energy per unit area, per unit time which is transported by the electromagnetic fields, while $\mu_0\varepsilon_0\mathbf{S}$ is the momentum per unit volume stored in the fields. The stress tensor $\overleftrightarrow{\boldsymbol{T}}$ is the electromagnetic stress of force per unit area acting on a surface of a volume, while $-\overleftrightarrow{\boldsymbol{T}}$ describes the flow of momentum or momentum current density transported by the fields.

The fields thus carry energy

$$\boldsymbol{u}_{em} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right),$$

momentum

$$\wp_{em} = \epsilon_0\mu_0\mathbf{S} = \epsilon_0(\mathbf{E}\times\mathbf{B}),$$

and even angular momentum

$$\ell_{em} = \mathbf{r}\times\wp_{em} = \epsilon_0(\mathbf{r}\times\mathbf{E}\times\mathbf{B})$$

This is even true for static fields, as long as **E** x **B** is nonzero.