1)

Consider electromagnetic waves in free space of the form

$\mathbf{E}(x,y,z,t) = \mathbf{E}_0(x,y)e^{ikz-i\omega t}$

$\mathbf{B}(x,y,z,t) = \mathbf{B}_0(x,y)e^{ikz-i\omega t}$

where $\mathbf{E}_0$ and $\mathbf{B}_0$ are in the xy plane.

a) Find the relation between $\boldsymbol{k}$ and ω, as well as the relation between $\mathbf{E}_0(x,y)$ and $\mathbf{B}_0(x,y)$.

b) Show that $\mathbf{E}_0(x,y)$ and $\mathbf{B}_0(x,y)$ satisfy the equations for electrostatics in free space.

2)

a) Write down Maxwell's equations assuming that no dielectrics or magnetic materials are present. In all of the following you must justify your answer.

b) If the signs of all the source charges are reversed, what happens to the electric and magnetic fields **E** and **B**?

c) If the system is space inverted, i.e., $\mathbf{x} \rightarrow \mathbf{x}' = -\mathbf{x}$, what happens to the charge density and current density, ρ and **j**, and to **E** and **B**?

d) If the system is time reversed, i.e., $t \rightarrow t' = -t$, what happens to ρ, **j**, **E** and **B**?

3)

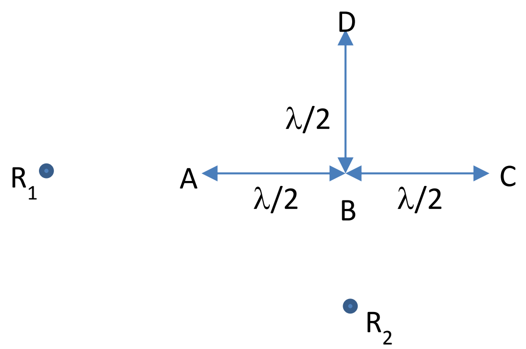
Four identical coherent monochromatic wave sources A, B, C, D, as shown in the figure below produce waves of the same wavelength λ. Two receivers R_1 and R_2 are at great (but equal) distances from B.

a) Which receiver picks up the greater signal?

b) Which receiver, if any, picks up the greater signal if source B is turned off?

c) If source D is turned off?

d) Which receiver can tell which source, B or D, has been turned off?



Solutions 1.

a) In case of a classical harmonic oscillator, one would take the solution, e.g. $A(t)= A_0 \sin\omega t$, and plug it back into the second-order equation of motion. This supplies an expression for the period of the oscillation, which depends on ω.
In the present case, the equations of motion are represented by the Maxwell's equations and we can try to plug in the wave equations into the Faraday and Ampere laws in vacuum:

$$\nabla\times\mathbf{E}=\left[-ikE_{0y}\hat{x}+ikE_{0x}\hat{y}+\left(\frac{\partial E_{0y}}{\partial x}-\frac{\partial E_{0x}}{\partial y}\right)\hat{z}\right]e^{i(kz-\omega t)}$$
$$=\left[ik\hat{z}\times\mathbf{E}_0+\nabla\times\mathbf{E}_0\right]e^{i(kz-\omega t)}$$

A similar expression can be obtained for $\nabla\times\mathbf{B}$ and we can rewrite the Maxwell's equations

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$$

$$\nabla\times\mathbf{B}=-\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}$$

as

$$ik\hat{z}\times\mathbf{E}_0(x,y)=i\omega\mathbf{B}_0(x,y)-\nabla\times\mathbf{E}_0$$

$$ik\hat{z}\times\mathbf{B}_0(x,y)=-i\frac{\omega}{c^2}\mathbf{E}_0(x,y)-\nabla\times\mathbf{B}_0\,.$$

$\nabla\times\mathbf{E}_0$ and $\nabla\times\mathbf{B}_0$ have only z-components while $\hat{z}\times\mathbf{E}_0$ and $\hat{z}\times\mathbf{B}_0$ are in the *xy* plane. Therefore, we require $\nabla\times\mathbf{E}_0=0$ and $\nabla\times\mathbf{B}_0=0$.
Then

$$\hat{z}\times\mathbf{E}_0(x,y)=\frac{\omega}{k}\mathbf{B}_0(x,y) \qquad (1)$$

$$\hat{z}\times\mathbf{B}_0(x,y)=-\frac{\omega}{kc^2}\mathbf{E}_0(x,y)\,. \qquad (2)$$

(1) and (2) provide the relation between $\mathbf{E}_0(x,y)$ and $\mathbf{B}_0(x,y)$ and show that $\mathbf{E}_0$, $\mathbf{B}_0$ and $\hat{z}$ are mutually perpendicular forming a right-hand set. Their Amplitudes are related by

$$\left|\mathbf{E}_0(x,y)\right|=\frac{\omega}{k}\left|\mathbf{B}_0(x,y)\right|$$

Considering the anticommutativity of the vector product, we obtain from (1):

$$\mathbf{E}_0=-\frac{\omega}{k}\hat{z}\times\mathbf{B}_0$$

Substitution in (2) gives

$$\frac{\omega^2}{k^2c^2} = 1 \quad \text{or} \quad k = \frac{\omega}{c}.$$

b) We used already the Ampere and Faraday laws to derive our results. Using $\nabla\mathbf{E} = 0$ and $\nabla\mathbf{B} = 0$ we find that $\nabla\mathbf{E}_0 = 0$ and $\nabla\mathbf{B}_0 = 0$. $\mathbf{E}_0(x,y)$ and $\mathbf{B}_0(x,y)$ therefore satisfy the equations for electro- and magnetostatics in free space.

Solutions 2.

a) In the absence of dielectric or magnetic materials Maxwell's equations are

$$\begin{cases} \nabla\cdot\mathbf{E} = \rho/\varepsilon_0 \\ \nabla\times\mathbf{E} = -\dfrac{\partial\mathbf{B}}{\partial t} \\ \nabla\cdot\mathbf{B} = 0 \\ \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\varepsilon_0\dfrac{\partial\mathbf{E}}{\partial t} \end{cases}$$

b) Under charge conjugation $e \to -e$, we have $\nabla \to \nabla' = \nabla$, $\dfrac{\partial}{\partial t} \to \dfrac{\partial}{\partial t'} = \dfrac{\partial}{\partial t}$, $\rho \to \rho' = -\rho$ and $\mathbf{J} \to \mathbf{J}' = -\mathbf{J}$. Under this transformation Maxwell's equations remain the same:

$$\begin{cases} \nabla'\cdot\mathbf{E} = \rho'/\varepsilon_0 \\ \nabla'\times\mathbf{E}' = -\dfrac{\partial\mathbf{B}'}{\partial t'} \\ \nabla'\cdot\mathbf{B}' = 0 \\ \nabla'\times\mathbf{B}' = \mu_0\mathbf{J}' + \mu_0\varepsilon_0\dfrac{\partial\mathbf{E}'}{\partial t'} \end{cases}$$

Comparing this with the first equations in (a), we see that, as $\rho' = -\rho$, $\mathbf{E}(\mathbf{r},t) = -\mathbf{E}(\mathbf{r},t)$.

Substituting this in the Ampere-Maxwell law in (a), we see that

$$\nabla'\times\mathbf{B}' = \nabla\times\mathbf{B}' = -\mu_0\mathbf{J} - \mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

Hence

$\mathbf{B}'(\mathbf{r},t) = -\mathbf{B}(\mathbf{r},t)$.

c) Under space inversion $\mathbf{r} \to \mathbf{r}' = -\mathbf{r}$, $\nabla \to \nabla' = -\nabla$, $\dfrac{\partial}{\partial t} \to \dfrac{\partial}{\partial t'} = \dfrac{\partial}{\partial t'}$, $e \to e' = e$.

Then

$\rho(\mathbf{r},t) \to \rho'(\mathbf{r},t) = \rho$

$\mathbf{j} \to \mathbf{j}' = \rho'\mathbf{u}' = -\rho\mathbf{u} = -\mathbf{j}$, where $\mathbf{u}$ is the velocity of the charges in an elementary volume.

As Maxwell's equations remain the same under this transformation we have

$\mathbf{E}'(\mathbf{r},t) = -\mathbf{E}(\mathbf{r},t)$

$\mathbf{B}'(\mathbf{r},t) = \mathbf{B}(\mathbf{r},t).$

d) Under time reversal, $\frac{\partial}{\partial t} \to \frac{\partial}{\partial t'} = -\frac{\partial}{\partial t'}$, $\nabla \to \nabla' = \nabla$, $e \to e' = e$.

Then $\rho' = \rho$, $\mathbf{j} \to \mathbf{j}' = -\rho\mathbf{u} = -\mathbf{j}$ and we have from the covariance of Maxwell's equation that

$\mathbf{E}'(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t)$

$\mathbf{B}'(\mathbf{r},t) = -\mathbf{B}(\mathbf{r},t).$

Solutions 3.

(a) Let r be the distance of R_1 and R_2 from B. We are given $r \gg \lambda$. Suppose the amplitude of the electric vector of the electromagnetic waves emitted by each source is E_0. The total amplitudes of the electric field at the receivers are

$$R_1: \quad E_{10} = E_0 \exp\left[iK\left(r - \frac{\lambda}{2}\right)\right] + E_0 e^{iKr} + E_0 \exp\left[iK\left(r + \frac{\lambda}{2}\right)\right] + E_0 \exp\left[iK\sqrt{r^2 + \frac{\lambda^2}{4}}\right],$$

$$R_2: \quad E_{20} = E_0 e^{iKr} + E_0 \exp\left[iK\left(r + \frac{\lambda}{2}\right)\right] + 2E_0 \exp\left[iK\sqrt{r^2 + \frac{\lambda^2}{4}}\right].$$

As $K\lambda = \frac{2\pi\nu\lambda}{c} = 2\pi$, $\exp[\pm i\frac{K\lambda}{2}] = e^{\pm i\pi} = -1$. With $r \gg \lambda$, $\sqrt{r^2 + \frac{\lambda^2}{4}} \approx r$. Thus

$$E_0 \exp\left[iK\sqrt{r^2 + \frac{\lambda^2}{4}}\right] \approx E_0 e^{iKr},$$

and we have

$$E_{10} \approx 0, \qquad E_{20} \approx 2E_0 e^{iKr}.$$

The intensity of a signal $I \propto |E|^2$, so the intensities received by R_1 and R_2 are respectively

$$I_0 = 0\,, \qquad I_2 \sim 4E_0^2\,.$$

Hence R_2 picks up greater signal.

(b) If source B is turned off, then

$$E_{10} \approx -E_0 e^{iKr}\,, \qquad E_{20} \approx E_0 e^{iKr}\,.$$

Thus $I_1 = I_2 \sim E_{10}^2$, that is, the two receivers pick up signals of the same intensity.

(c) If source D is turned off, one has

$$E_{10} \approx -E_0 e^{iKr}\,, \qquad E_{20} \approx 3E_0 e^{iKr}\,,$$

and

$$I_1 \sim E_{10}^2\,, \qquad I_2 \sim 9E_0^2\,.$$

Hence R_2 picks up greater signal.

(d) From the above, we can see that I_1 remains the same whether the sources B and D are on or off. Hence R_1 cannot determine the on–off state of B and D. On the other hand, the intensity of I_2 differs for the three cases above so the strength of the signal received by R_2 can determine the on–off state of the sources B and D.