Solving Laplace's Equation inside Matter

- Inside matter, we still have $\nabla \times \mathbf{E} = \mathbf{0}$
- Hence the electric potential *V* is still well-defined.
- However, the Poisson's equation now becomes

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\left(\rho_f - \nabla \cdot \mathbf{P}\right) \Rightarrow \nabla^2 V = -\frac{1}{\varepsilon_0}\left(\rho_f - \nabla \cdot \mathbf{P}\right)$$

- **P** must be given in order for one to solve E
- For example, for “frozen-in” polarization, the exact form of P may be given, or the relation of P to E (called the constitutive equation) should be given.

- Again, to avoid solving the Poisson's equation inside a region, we shall only consider systems in which the right hand side of

$$\nabla^2 V = -\frac{1}{\varepsilon_0}\left(\rho_f - \nabla \cdot \mathbf{P}\right)$$

is zero except on some very thin layers
- For this to hold, obviously we need to assume $\rho_f = 0$
- Hence we shall only consider systems with no volume free charge.

- In addition, we also need to have

$$\nabla \cdot \mathbf{P} = 0$$

- This holds under two situations:
- When we have uniform "frozen-in" polarization.
- When the media are linear dielectrics, then $\rho_f = 0 \rightarrow \rho_b = 0$.

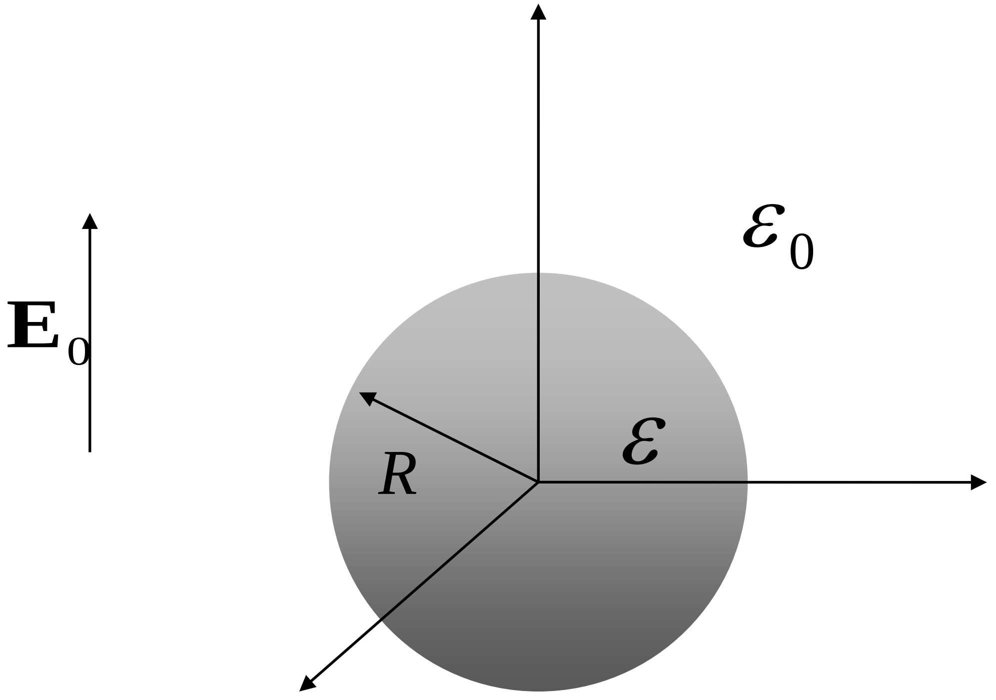
- Under these conditions, we only need to solve the Laplace’s equation in different regions.
- Across boundaries, the differentials equations $\nabla \times \mathbf{E} = \mathbf{0}$ and $\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\left(\rho_f - \nabla \cdot \mathbf{P}\right)$

  are transformed to their integral forms to obtain the boundary conditions

$$\mathbf{E}^{//}_{\text{above}} = \mathbf{E}^{//}_{\text{below}} \qquad D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = \sigma_f$$

Example:

A sphere of linear dielectric material is placed in an originally uniform electric field $\mathbf{E}_0$. Find the new field inside the sphere.



Solution:

We know that $V \to -E_0 r\cos\theta$ as $r \to \infty$

Since there are no free charges,

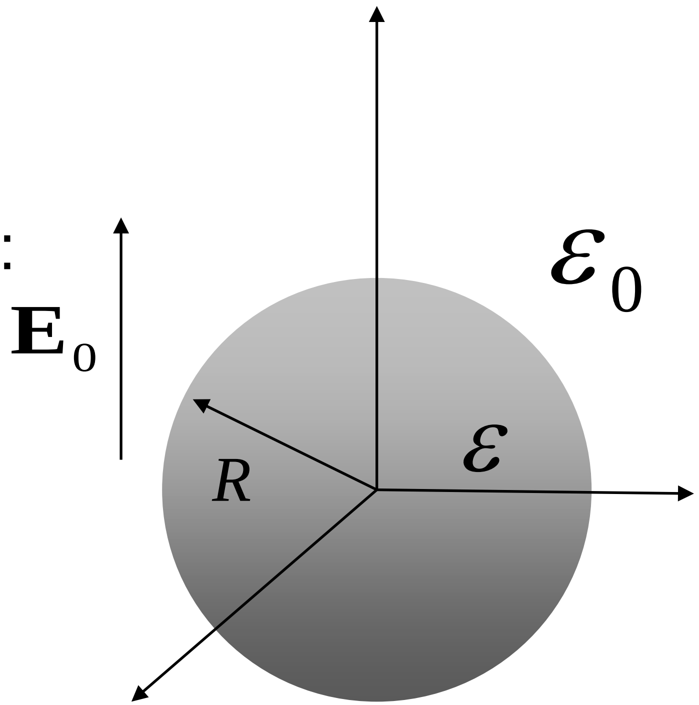
$$\nabla \cdot \mathbf{D} = 0$$

both inside and outside the sphere:

$$\therefore \begin{cases} \nabla \cdot (\varepsilon\, \mathbf{E}) = 0 & \text{for } r < R \\ \nabla \cdot (\varepsilon_0 \mathbf{E}) = 0 & \text{for } r > R \end{cases}$$

$$\therefore \nabla \cdot \mathbf{E} = 0$$

both inside and outside the sphere



Solution:

$$\nabla \cdot \mathbf{E} = 0$$

$$\therefore \nabla^2 V = 0$$

Therefore

$$\begin{cases} V_{\text{out}}(r,\theta) = -E_0 r\cos\theta + \sum_{l=1}^{\infty} \dfrac{A_l}{r^{l+1}} P_l(\cos\theta) \\ V_{\text{in}}(r,\theta) = \sum_{l=0}^{\infty} B_l r^l P_l(\cos\theta) \end{cases}$$

(*Note that* $A_0 = 0$ *since there is no net charge*)

The boundary conditions are:

1) $V_{\text{in}}\left(R,\theta\right)=V_{\text{out}}\left(R,\theta\right)$

i.e.

$$-E_0 R\cos\theta+\sum_{l=1}^{\infty}\frac{A_l}{R^{l+1}}P_l(\cos\theta)=\sum_{l=0}^{\infty}B_l R^l P_l(\cos\theta)$$

$$\therefore \quad \begin{cases} B_0=0 \\ B_1=-E_0+\dfrac{A_1}{R^3} \\ B_l=\dfrac{A_l}{R^{2l+1}} \qquad l=2,3,4,\ldots \end{cases}$$

The boundary conditions are:

2) The normal component of **D** being continuous at *R* implies

$$\varepsilon_0 \left.\frac{\partial V_{\text{out}}}{\partial r}\right|_R = \varepsilon \left.\frac{\partial V_{\text{in}}}{\partial r}\right|_R$$

i.e.

$$-\varepsilon_0 E_0 \cos\theta - \varepsilon_0 \sum_{l=1}^{\infty} (l+1) \frac{A_l}{R^{l+2}} P_l(\cos\theta) = \varepsilon \sum_{l=1}^{\infty} l B_l R^{l-1} P_l(\cos\theta)$$

$$\therefore \begin{cases} \varepsilon B_1 = -\varepsilon_0 E_0 - \dfrac{2\varepsilon_0 A_1}{R^3} \\ \varepsilon B_l = -\varepsilon_0 \dfrac{l+1}{l} \dfrac{A_l}{R^{2l+1}} \qquad l = 2, 3, 4, \ldots \end{cases}$$

For $l = 1$,

$$\begin{cases} B_1 = -E_0 + \dfrac{A_1}{R^3} \\ \varepsilon B_1 = -\varepsilon_0 E_0 - \dfrac{2\varepsilon_0 A_1}{R^3} \end{cases}$$

$$\Rightarrow \begin{cases} A_1 = \dfrac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} E_0 R^3 \\ B_1 = -\dfrac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0} E_0 \end{cases}$$

For $l = 2,3,4,\ldots$

$$\begin{cases} B_l = \dfrac{A_l}{R^{2l+1}} \\ \varepsilon B_l = -\varepsilon_0 \dfrac{l+1}{l} \dfrac{A_l}{R^{2l+1}} \end{cases}$$

$$\Rightarrow A_l = B_l = 0$$

In conclusion,

$$\begin{cases} A_1 = \dfrac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} E_0 R^3 \\ B_1 = -\dfrac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0} E_0 \\ A_l = B_l = 0 \qquad \text{for } l = 0, 2, 3, 4, \ldots \end{cases}$$

Therefore,

$$\begin{cases} V_{\text{in}}(r,\theta) = -\dfrac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0} E_0 r\cos\theta \\ V_{\text{out}}(r,\theta) = -E_0 r\cos\theta + \dfrac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} E_0 R^3 \dfrac{\cos\theta}{r^2} \end{cases}$$

So,

$$\mathbf{E}_{\text{in}} = -\nabla V_{\text{in}} = \frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0} E_0 \hat{\mathbf{z}} = \frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0} \mathbf{E}_0$$

$$\begin{cases} V_{\text{in}}(r,\theta) = -\dfrac{3\varepsilon_0}{\varepsilon+2\varepsilon_0} E_0 r\cos\theta \\ V_{\text{out}}(r,\theta) = -E_0 r\cos\theta + \dfrac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0} E_0 R^3 \dfrac{\cos\theta}{r^2} \end{cases}$$

$$\because \quad \mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}_{\text{in}} = (\varepsilon-\varepsilon_0)\mathbf{E}_{\text{in}} \qquad \varepsilon = \varepsilon_0(1+\chi_e)$$

$$\mathbf{P} = \frac{3\varepsilon_0(\varepsilon-\varepsilon_0)}{\varepsilon+2\varepsilon_0}\mathbf{E}_0$$

$$\mathbf{P}\times\frac{4\pi}{3}R^3 = 4\pi R^3 \frac{\varepsilon_0(\varepsilon-\varepsilon_0)}{\varepsilon+2\varepsilon_0}\mathbf{E}_0 = \mathbf{p}$$

V_{out} is equivalent to that due to the external field $\mathbf{E}_0$ and a dipole $\mathbf{p} = 4\pi R^3 \dfrac{\varepsilon_0(\varepsilon-\varepsilon_0)}{\varepsilon+2\varepsilon_0}\mathbf{E}_0$ at the center.

$$V_{\text{out}} = -E_0 r\cos\theta + \frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0} E_0 R^3 \frac{\cos\theta}{r^2} = -E_0 r\cos\theta + \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$

= External field potential

+ potential due to a pure dipole $\mathbf{p}$ at the center.

$$\mathbf{E}_{\text{in}} = \frac{3\varepsilon_0}{\varepsilon+2\varepsilon_0}\mathbf{E}_0$$

$$= \mathbf{E}_0 + \frac{\varepsilon_0-\varepsilon}{\varepsilon+2\varepsilon_0}\mathbf{E}_0$$

$$= \mathbf{E}_0 - \frac{1}{3\varepsilon_0}\mathbf{P}$$ = External field + Field due to $\mathbf{P}$

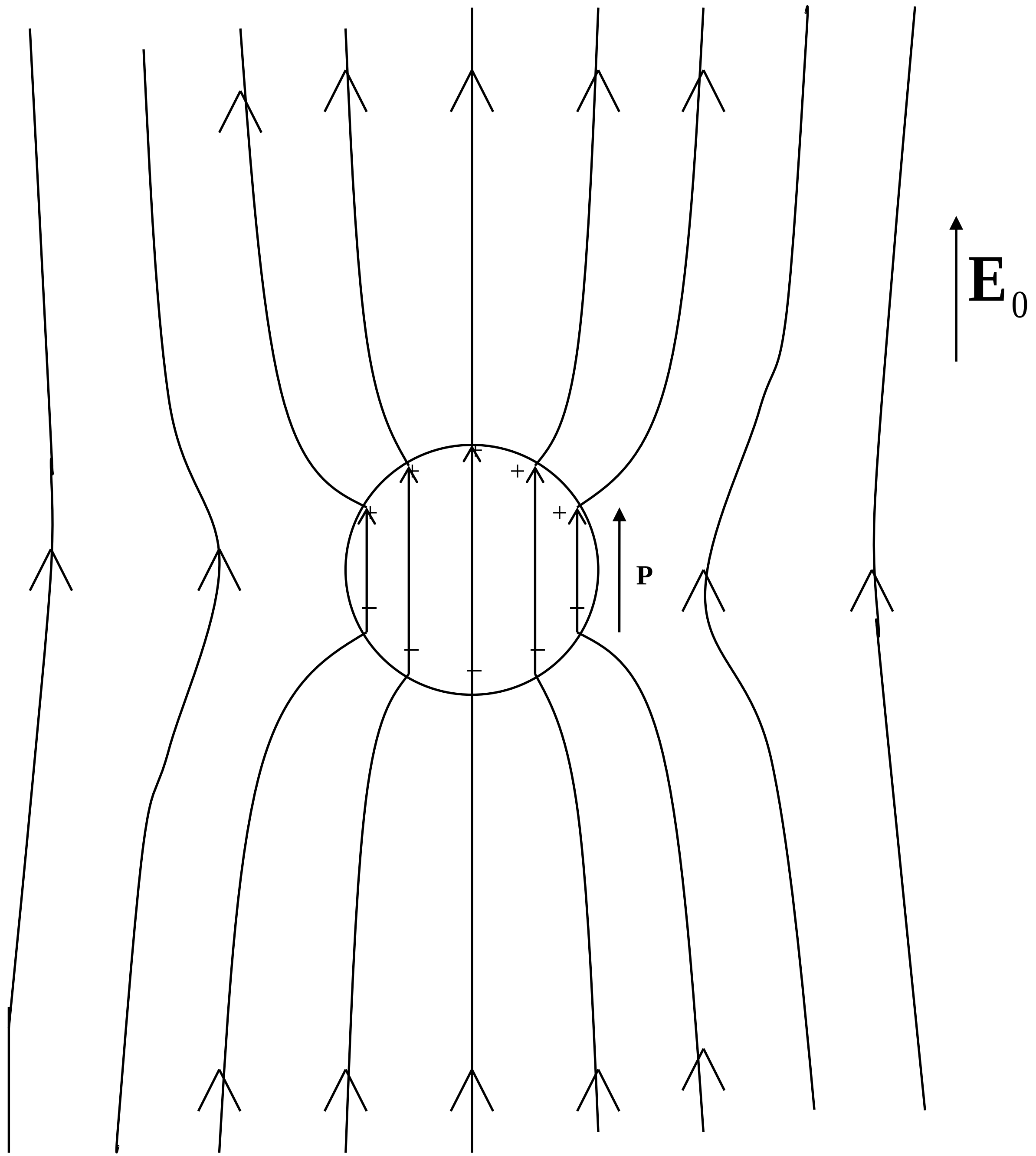
There are no volume bound charges since

$$\rho_b = -\nabla \cdot \mathbf{P} = 0$$

The surface bound charge density is

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{r}}$$

$$= \frac{3\varepsilon_0(\varepsilon - \varepsilon_0)}{\varepsilon + 2\varepsilon_0} E_0 \cos\theta$$



$\mathbf{E}_0$
$\mathbf{P}$