Electrodynamics and the Maxwell's equations

Summary:

Electrostatics and Magnetostatics

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$$ ← *Gauss' Law*

$$\nabla \times \mathbf{E} = 0$$ ← *No name*

$$\nabla \cdot \mathbf{B} = 0$$ ← *No name*

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$ ← *Ampere's Law*

Summary of Chapter 2-6:

Electrostatics and Magnetostatics

Inside matter:

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\ \nabla \times \mathbf{E} = 0 \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J}_f \end{cases}$$

where

$$\begin{cases} \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} = \dfrac{1}{\mu_0} \mathbf{B} - \mathbf{M} \end{cases}$$

Summary of Chapter 2-6:

Electrostatics and Magnetostatics

where

$$\begin{cases} \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} = \dfrac{1}{\mu_0} \mathbf{B} - \mathbf{M} \end{cases}$$

For instance, in linear media,

$$\begin{cases} \mathbf{D} = \varepsilon \mathbf{E} \\ \mathbf{H} = \dfrac{1}{\mu} \mathbf{B} \end{cases}$$

For the set of equations to be closed,

one has to supply the relation between **D, E** *and* **H, M,**

which are called the constitutive relations.

Summary of Chapter 2-6:

Electrostatics and Magnetostatics

The force a charge q moving with velocity $\mathbf{v}$ experiences in a region of E field and B field is given by the Lorentz force law:

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

Summary of Chapter 2-6:

Electrostatics and Magnetostatics

In the static cases,
the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

AUTOMATICALLY SATISFIED!!!

Since:

However

(1) $$\frac{\partial \rho}{\partial t} = 0$$

(2) $$\nabla \cdot \mathbf{J} = \frac{1}{\mu_0} \nabla \cdot \left(\nabla \times \mathbf{B} \right) = 0$$

the two curl equations have to be modified in electrodynamics

Electromagnetic induction

Faraday's Experiments

- In the 19th century, Faraday performed a series of experiments which showed that **in general**, the electric field is **not curl-free**.

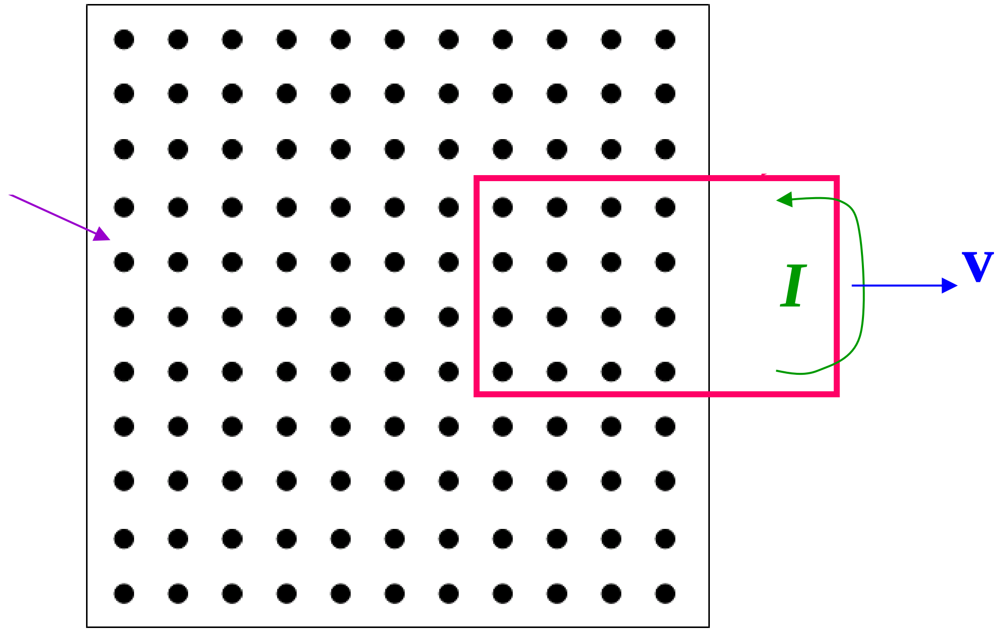
Experiment 1:
A loop of wire partly inside a magnetic field (assume uniform for simplicity) moving with velocity ***v*** *perpendicular to the field.*

Experiment 2:
A magnetic field partly inside a loop of wire moving to the opposite direction.

Experiment 3:
A loop at rest inside a changing magnetic field.

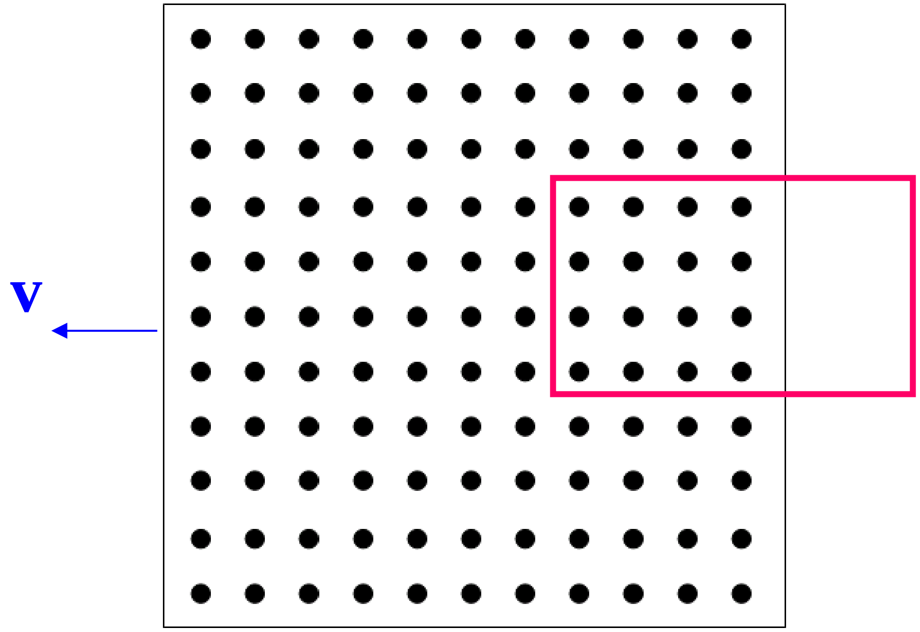
Experiment 1:

- A loop of wire partly inside a magnetic field (assume uniform for simplicity) moving with velocity **v** perpendicular to the field.



Experiment 2:

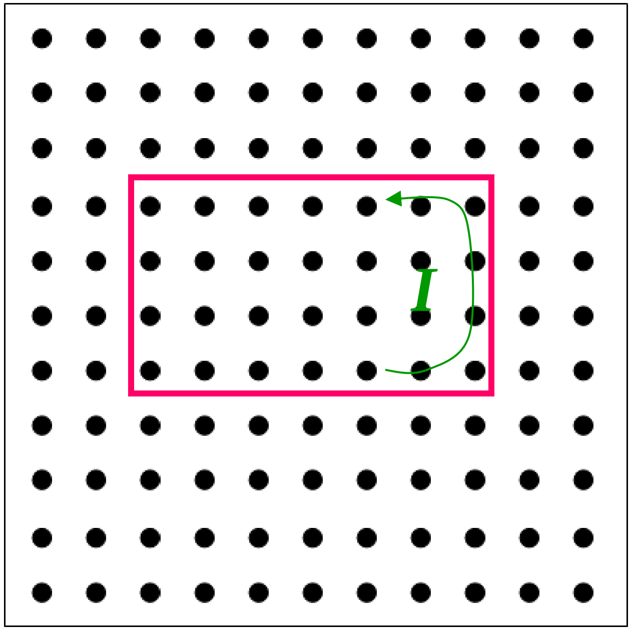
- A magnetic field partly inside a loop of wire moving to the opposite direction.

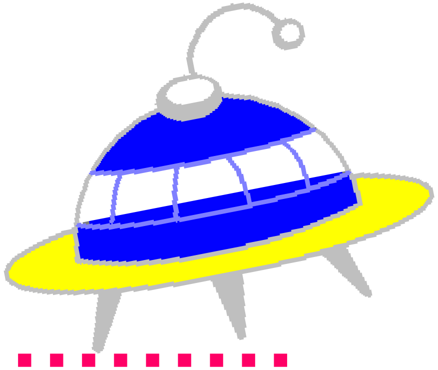


What can we observe in this experiment?

Experiment 3:

- A loop at rest inside a changing magnetic field.





charging B-field…………

What is the conclusion in the 3 experiments?

Observation

- In all the experiments, there will be a **current** flowing.
- There is a current because there is a force driving the charges to move.

Let $\mathbf{f}$ be the force per unit charge.
The electromotive force (emf) $\mathcal{E}$ is defined by

$$\varepsilon = \oint \mathrm{f} \cdot d\mathrm{l}$$

over a closed loop.

Observation

- There is a current because there is a force driving the charges to move.

$$\varepsilon = \oint \mathbf{f} \cdot d\mathbf{l}$$

- When there is a driving force, it is a "rule of thumb" that a current will be generated which is proportional to **f**:

$$\mathbf{J} = \sigma \mathbf{f}$$

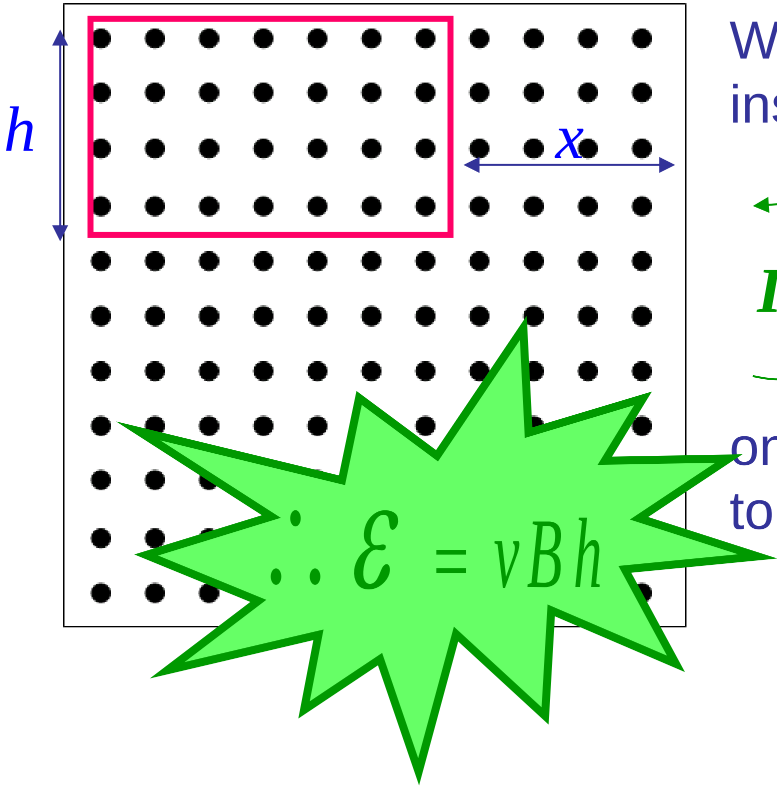
conductivity of the material,

where $\rho = \dfrac{1}{\sigma}$*, is called the resistivity*

- The source of this driving force in the Faraday's experiments has different interpretations though.

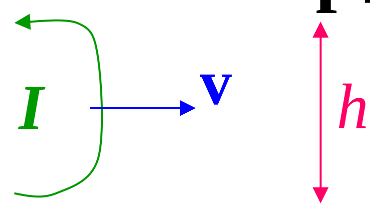
Experiment 1:

- The force is due to the Lorentz force of charges in motion → Motional emf.



When the loop moves, the charges inside experience a force

$$\mathbf{f} = \mathbf{v} \times \mathbf{B}$$



(pointing upward with magnitude vB)

only the left side of the loop contributes to the emf

$$\varepsilon = \oint \mathbf{f} \cdot d\mathbf{l}$$

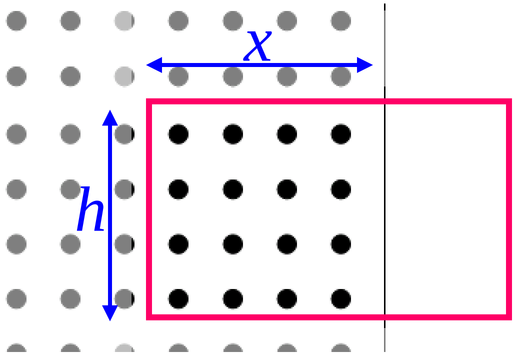
(counterclockwise as positive)

Experiment 1:

- The force is due to the Lorentz force of charges in motion → Motional emf.
- Notice that the emf in this case can be related to the magnetic flux through the loop.

$$\Phi = \int \mathbf{B} \cdot d\mathbf{a}$$ ***(inwards as positive)***

- The sign convention of emf and flux has to be consistent by right hand rule.



In this particular case, obviously

$$\Phi = Bhx$$

(where x is the portion of the length of the loop inside the field.)

Hence

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

The relation is hence

$$\frac{d\Phi}{dt} = Bh\frac{dx}{dt} = -vBh$$

which is called the flux rule ← ***valid in general for a loop moving in a non-uniform B-field***

Experiment 2,3:

Imagine an observer in experiment 1 moving with velocity $\mathbf{v}$.

What he will observe is exactly that in experiment 2 there is a loop at rest with a magnetic field moving to the right.

- A current and hence electromotive force will still be observed.
- there should be no Lorentz force due to magnetic field since the loop is not moving.
- it can be concluded that there is an electric field

Faraday's law:

- *Faraday proposed that a changing magnetic field will induce an electric field.*

The flux rule is still correct.

However, this time the driving force is due to an induced electric field.

Hence

$$\frac{d\Phi}{dt} = -\mathcal{E}$$

$$\boxed{\frac{d}{dt}\int \mathbf{B}\cdot d\mathbf{a} = -\oint \mathbf{E}\cdot d\mathbf{l}} \qquad \text{(Faraday's law in integral form)}$$

$$\Rightarrow \int \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{a} = -\int (\nabla\times\mathbf{E})\cdot d\mathbf{a}$$

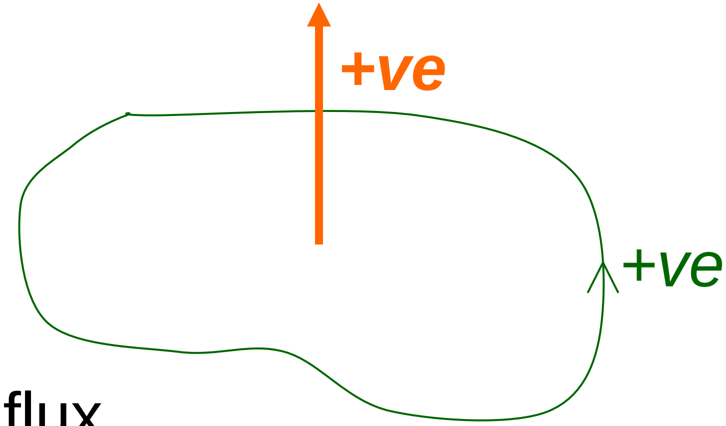
$$\Leftrightarrow \boxed{\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}} \qquad \text{(Faraday's law in differential form)}$$

Faraday's law:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Note that the minus sign denotes what is called the *Lenz's law* :

Nature abhors a change in flux!



e.g., Flux increases
→ negative $\nabla \times \mathbf{E}$
→ negative current
→ produces negative flux
→ opposes the change in flux

Faraday’s law:

- The induced electric field forms closed loops and is divergence free.
- Therefore, the total electric field due to charges and changing magnetic field satisfies

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0, \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Conclusion

$$\begin{cases} \nabla\cdot\mathbf{E}=\rho/\varepsilon_0 & \text{Gauss' law} \\ \nabla\times\mathbf{E}=-\dfrac{\partial\mathbf{B}}{\partial t} & \text{Faraday's law} \\ \nabla\cdot\mathbf{B}=0 & \text{No name} \\ \nabla\times\mathbf{B}=\mu_0\mathbf{J} & \text{Ampere's law} \end{cases}$$

Maxwell's Correction

- With the Faraday's law, the set of equations now reads

$$\begin{cases} \nabla \cdot \mathbf{E} = \rho / \varepsilon_0 \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \end{cases}$$

Maxwell's Correction

If you study them carefully, you will realize that something is wrong!!

- Look at the fourth equation, and take divergence of both sides:

$$\nabla \cdot \mathbf{J} = \frac{1}{\mu_0} \nabla \cdot \left(\nabla \times \mathbf{B} \right) = 0$$

- However, from the continuity equation:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

which is in general non-zero in electrodynamics.

Maxwell's Correction

- In addition, consider the Ampere's law in integral form:

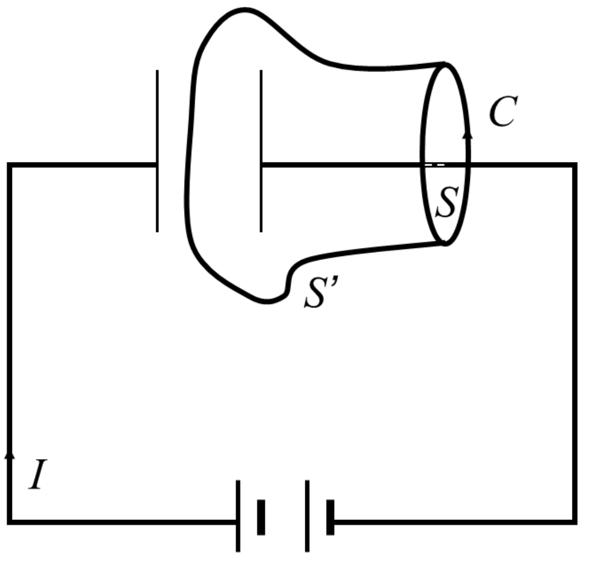
$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{a} = \mu_0 I_{\text{enc}}$$

- The current enclosed by C is not well defined since different choices of S may yield different I_{enc}
- This is, of course, also due to the fact that $\nabla \cdot \mathbf{J} \neq 0$ in general.

Maxwell's Correction

Consider the following set up of charging up a capacitor:

- When the capacitor is being charged up, a current is flowing in the direction shown
- Positive and negative charges are being accumulated on the left and right plate of the capacitor, respectively.



- In between the plates, the electric field is increasing, but there is no current.

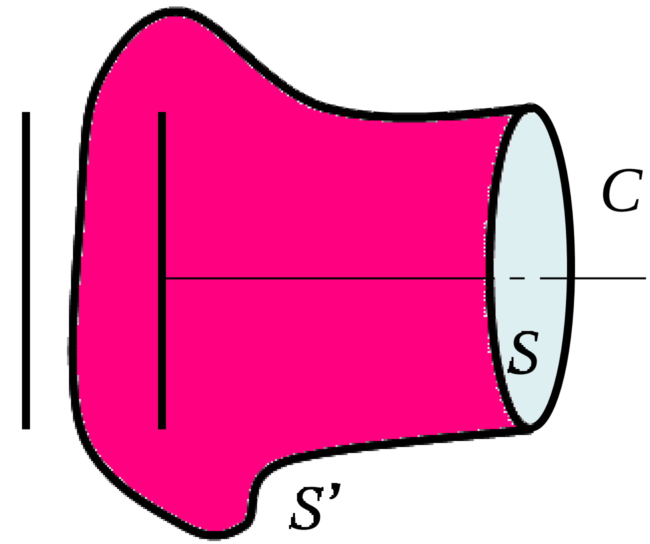
Maxwell's Correction

Consider the amperian loop *C*, which is assumed to be "flat" for simplicity. If Ampere's law is applied on the loop, and the flat surface S is used to calculate I_{enc} one obtains

$$I_{\text{enc}} = I$$

However, if the curved surface S' is chosen, which does not intersect with the wire, then

$$I_{\text{enc}} = 0$$



Maxwell's Correction

Hence, we know that something is missing on the right hand side of the Ampere's law, which, together with $\mu_0 \mathbf{J}$, gives a zero divergence.

Notice that from the continuity equation and Gauss' law:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}\left(\varepsilon_0 \nabla \cdot \mathbf{E}\right) = -\nabla \cdot \left(\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)$$

$$\Rightarrow \nabla \cdot \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right) = 0$$

Maxwell's Correction

The second term is sometimes called the displacement current:

$$\mathbf{J}_d = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Though it is misleading since it has nothing to do with flowing charges.

$$\mu_0 \mathbf{J}_d = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Maxwell proposed that the missing term in the Ampere's law is

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\mathbf{enc}} + \mu_0 \varepsilon_0 \int \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{a}$$

Maxwell's correction terms

integral form

Maxwell's Correction

- By adding this *"maxwell's correction term"*, the conservation of charges is restored.
- The ambiguity in the definition of current enclosed is also solved by including the *displacement current.*
- It turns out that it is the *sum of real current and displacement current* that is *unchanged* no matter what surface one chooses.
- Also note the parallelity between the modified Ampere's law and the Faraday's law,

A changing magnetic field induces an electric field

A changing electric field induces a magnetic field

Maxwell's Correction

- Hence there are two sources of magnetic field, viz.,

$$\mathbf{J} \text{ and } \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

- The second contribution $\mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ is difficult to observe as

$$\mu_0 \varepsilon_0 \approx 10^{-17}$$

  which is very small, unless the electric field is changing very rapidly.
- Maxwell derived this term relying solely on mathematics.
- It was later verified experimentally by the observation of electromagnetic waves.

Maxwell's Equations

The set of four equations now becomes

$$\begin{cases} \nabla \cdot \mathbf{E} = \rho / \varepsilon_0 & \text{Gauss' law} \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} & \text{Faraday's law} \\ \nabla \cdot \mathbf{B} = 0 & \text{No name} \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} & \text{Ampere's law with Maxwell's correction} \end{cases}$$

Electromagnetic Waves in Vacuum

- The Maxwell's equations predict the existence of electromagnetic waves.
- In vacuum, the Maxwell's equations read

$$\begin{cases} \nabla \cdot \mathbf{E} = 0 \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} \end{cases}$$

Electromagnetic Waves in Vacuum

Taking the curl on both sides of the Faraday's law, we have

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t} \nabla \times \mathbf{B}$$

By the Ampere's law,

$$\nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{\partial}{\partial t} \left(\mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

By Gauss' law

$$\nabla^2 \mathbf{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

wave equation
in 3D

Electromagnetic Waves in Vacuum

Similarly, by taking the curl on both sides of the Ampere's law, we have

$$\nabla \times (\nabla \times \mathbf{B}) = \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \nabla \times \mathbf{E}$$

By Faraday's law

$$\nabla (\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = -\mu_0 \varepsilon_0 \frac{\partial}{\partial t} \frac{\partial \mathbf{B}}{\partial t}$$

Since

$$\nabla \cdot \mathbf{B} = 0$$

hence

$$\nabla^2 \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

Electromagnetic Waves in Vacuum

Therefore, both the E field and B field satisfy the wave equation and admit solution of propagating waves.

cf. $$\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$ → speed of EM wave

$$\begin{aligned} c &= \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \\ &= \frac{1}{\sqrt{4\pi \times 10^{-7} \times 8.85 \times 10^{-12}}} \\ &= \frac{1}{\sqrt{1.11 \times 10^{-17}}} \\ &= 3.00 \times 10^{8}\,\mathrm{ms}^{-1} \end{aligned}$$

Maxwell's Equations Inside Matter

- Inside matter, there are in general polarization $\mathbf{P}$ and magnetization $\mathbf{M}$.
- The Gauss' law and the Ampere's law can be re-formulated.
- For the Gauss' law, the total charge is the sum of free charges and bound charges:

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\left(\rho_f + \rho_b\right)$$,where $\rho_b = -\nabla \cdot \mathbf{P}$

Hence $$\nabla \cdot \mathbf{D} = \rho_f$$,where $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$

Maxwell's Equations Inside Matter

- In magnetostatics, we have also learned that on the right hand side of the Ampere's law,
- the total current consists of two contributions, viz., free currents and bound currents due to magnetization.
- Hence, you may propose that the Ampere's law in electrodynamics should be

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J}_f + \mathbf{J}_b \right) + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

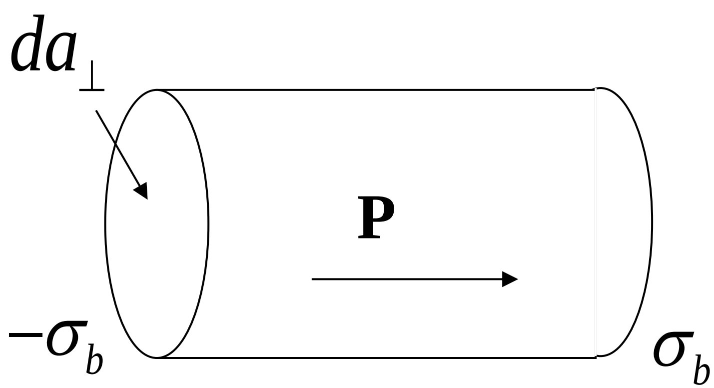
where $\mathbf{J}_b = \nabla \times \mathbf{M}$

Maxwell's Equations Inside Matter

- However, in electrodynamics, there is another contribution to the total current that we missed in the above equation.
- This means that the charges inside the electric dipoles are moving, giving rise to a current which is called the **polarization current** $\mathbf{J}_p$
- In electrodynamics, $\mathbf{P}$ varies with time in general.

Maxwell's Equations Inside Matter

Consider a small piece of matter with polarization **P**, as shown below:



We know that there will be surface bound charges at both ends of density

$$\sigma_b = P$$

Maxwell's Equations Inside Matter

When $\mathbf{P}$ varies, the net effect is that a current dI is flowing in the direction of $\mathbf{P}$.

The magnitude of the current is

$$dI = \frac{\partial}{\partial t}\left(\sigma_b da_\perp\right)$$

Hence, the volume current density is

$$\mathbf{J}_p = \frac{dI}{da_\perp}\hat{\mathbf{P}} = \frac{\partial \sigma_b}{\partial t}\hat{\mathbf{P}} = \frac{\partial P}{\partial t}\hat{\mathbf{P}} = \frac{\partial \mathbf{P}}{\partial t}$$

Maxwell's Equations Inside Matter

Taking into account the polarization current, the Ampere's law inside matter should be

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J}_f + \mathbf{J}_b + \mathbf{J}_p \right) + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_f + \mu_0 \nabla \times \mathbf{M} + \mu_0 \frac{\partial \mathbf{P}}{\partial t} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \times \left(\frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \right) = \mathbf{J}_f + \frac{\partial \left(\varepsilon_0 \mathbf{E} + \mathbf{P} \right)}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$,where $\mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M}$

Maxwell's Equations Inside Matter

The two remaining equations

$$\begin{cases} \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \end{cases}$$

involve no source and are hence unchanged inside matter.

In conclusion, inside matter:

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = J_f + \dfrac{\partial \mathbf{D}}{\partial t} \end{cases}$$

Maxwell's Equations Inside Matter

The equations are providing the constitutive relations, which relate polarization to the E field and magnetization to the B field.

e.g., for linear media,

$$\begin{cases} \mathbf{D} = \varepsilon \mathbf{E} \\ \mathbf{H} = \dfrac{1}{\mu} \mathbf{B} \end{cases}$$

Electromagnetic Waves in Matter

- *Inside matter with no free charges and currents, the Maxwell's equations become*

$$\begin{cases} \nabla \cdot \mathbf{D} = 0 \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \dfrac{\partial \mathbf{D}}{\partial t} \end{cases}$$

- *If the medium is linear, then the equations reduce to*

$$\begin{cases} \nabla \cdot \mathbf{E} = 0 \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu\varepsilon \dfrac{\partial \mathbf{E}}{\partial t} \end{cases}$$

Notice that these are just the Maxwell's equations in vacuum under the transcription $\varepsilon_0 \to \varepsilon,\ \mu_0 \to \mu$

Electromagnetic Waves in Matter

Hence, the E field and B field satisfy the wave equation

$$\begin{cases} \nabla^2 \mathbf{E} = \mu\varepsilon \dfrac{\partial^2 \mathbf{E}}{\partial t^2} \\ \nabla^2 \mathbf{B} = \mu\varepsilon \dfrac{\partial^2 \mathbf{B}}{\partial t^2} \end{cases}$$

and the speed of light becomes

$$v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \Bigg/ \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} = \frac{c}{n}$$

Electromagnetic Waves in Matter

In other words, the speed of light in matter is reduced by a factor

$$n = \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}}$$

which is called the refractive index.

For most materials, $\mu \approx \mu_0$, and $\varepsilon > \varepsilon_0$

$$n = \sqrt{\frac{\varepsilon}{\varepsilon_0}} = \sqrt{K} > 1$$

K : dielectric constant

Hence $v < c$