To Show that $\oint_S \frac{\hat{\mathbf{r}}'}{|\mathbf{r}-\mathbf{r}'|} da' = \begin{cases} \frac{4\pi}{3}\mathbf{r} & \text{if } r < R \\ \frac{4\pi R^3}{3r^3}\mathbf{r} & \text{if } r > R \end{cases}$

Without loss of generality, we can let $\mathbf{r}$ be on the z-axis, viz, $\mathbf{r} = r\hat{\mathbf{z}}$. Then due to symmetry, we only have to evaluate the z-component:

$$\int \frac{\hat{\mathbf{r}}'}{|\mathbf{r}-\mathbf{r}'|} da' = \int \frac{\hat{\mathbf{z}}\cos\theta}{\left(r^2+R^2-2Rr\cos\theta\right)^{1/2}} da'$$

$$= \hat{\mathbf{z}}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta R^2 \sin\theta \frac{\cos\theta}{\left(r^2+R^2-2Rr\cos\theta\right)^{1/2}}$$

$$= 2\pi R^2 \hat{\mathbf{z}} \int_0^{\pi} \sin\theta \frac{\cos\theta}{\left(r^2+R^2-2Rr\cos\theta\right)^{1/2}} d\theta$$

$$= 2\pi R^2 \hat{\mathbf{z}} \int_{\pi}^{0} \frac{\cos\theta}{\left(r^2+R^2-2Rr\cos\theta\right)^{1/2}} d\left(\cos\theta\right)$$

$$= 2\pi R^2 \hat{\mathbf{z}} \int_{-1}^{1} \frac{\alpha}{\left(r^2+R^2-2Rr\alpha\right)^{1/2}} d\alpha \qquad \left(\alpha = \cos\theta\right)$$

$$= 2\pi R^2 \hat{\mathbf{z}} \int_{(R+r)^2}^{(R-r)^2} \frac{\frac{r^2+R^2-\beta}{2Rr}}{\beta^{1/2}} \frac{1}{-2Rr} d\beta \qquad \left(\beta = r^2+R^2-2Rr\alpha\right)$$

$$= -\frac{1}{4R^2r^2} 2\pi R^2 \hat{\mathbf{z}} \int_{(R+r)^2}^{(R-r)^2} \frac{r^2+R^2-\beta}{\beta^{1/2}} d\beta$$

$$= -\frac{\pi}{2r^2}\hat{\mathbf{z}} \left[\left(r^2+R^2\right) 2\beta^{1/2} - \frac{2}{3}\beta^{3/2} \right]_{(R+r)^2}^{(R-r)^2}$$

$$= \frac{\pi}{r^2}\hat{\mathbf{z}} \left[\frac{1}{3}\beta^{3/2} - \left(r^2+R^2\right)\beta^{1/2} \right]_{(R+r)^2}^{(R-r)^2}$$

If $r < R$

$$\int \frac{\hat{\mathbf{r}}'}{|\mathbf{r}-\mathbf{r}'|}da' = \frac{\pi}{r^2}\hat{\mathbf{z}}\left[\frac{1}{3}\left((R-r)^3-(R+r)^3\right)-\left(r^2+R^2\right)(R-r-R-r)\right]$$

$$= \frac{\pi}{r^2}\hat{\mathbf{z}}\left[\frac{1}{3}\left(R^3-3R^2r+3Rr^2-r^3-R^3-3R^2r-3Rr^2-r^3\right)+2r\left(r^2+R^2\right)\right]$$

$$= \frac{\pi}{r^2}\hat{\mathbf{z}}\left[\frac{1}{3}\left(-3R^2r-r^3-3R^2r-r^3\right)+2r\left(r^2+R^2\right)\right]$$

$$= \frac{\pi}{r^2}\hat{\mathbf{z}}\left[-\frac{2}{3}r^3+2r^3\right]$$

$$= \frac{4}{3}\pi r\hat{\mathbf{z}} = \frac{4}{3}\pi\mathbf{r}$$

If $r > R$

$$\int \frac{\hat{\mathbf{r}}'}{|\mathbf{r}-\mathbf{r}'|}da' = \frac{\pi}{r^2}\hat{\mathbf{z}}\left[\frac{1}{3}\left((r-R)^3-(R+r)^3\right)-\left(r^2+R^2\right)(r-R-R-r)\right]$$

$$= \frac{\pi}{r^2}\hat{\mathbf{z}}\left[\frac{1}{3}\left(r^3-3r^2R+3rR^2-R^3-R^3-3R^2r-3Rr^2-r^3\right)+2R\left(r^2+R^2\right)\right]$$

$$= \frac{2\pi}{r^2}\hat{\mathbf{z}}\left[-\frac{1}{3}R\left(3r^2+R^2\right)+R\left(r^2+R^2\right)\right]$$

$$= \frac{2\pi}{r^2}R\hat{\mathbf{z}}\left[\frac{2}{3}R^2\right]$$

$$= \frac{4\pi R^3}{3r^3}\mathbf{r}$$