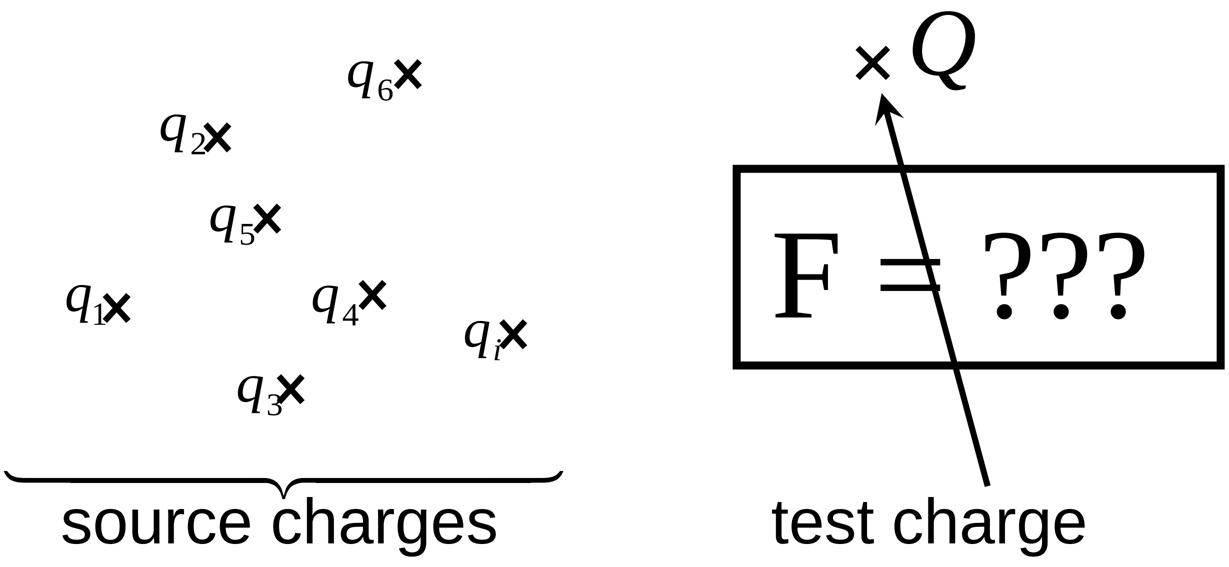
What is the force exerted on a test charge Q, by some source charges $q_1, q_2, q_3, \cdots$?



- When the source charges are at rest, it is observed that the force acting on the test charge is in general position dependent but independent of the motion of the test charge
- Hence one can
  - assigning to the test charge a number *Q*, called its charge
  - assigning to every point in space a vector called the electric field **E**
- The force can then be given by

$$\mathbf{F}_E = Q\mathbf{E}$$

- This is called the electric force

What if the source charges are moving?

- The law

$$\mathbf{F} = m\mathbf{a} = Q\mathbf{E}$$

where **F** depends on position only, but independent of the motion of the test charge, is inconsistent with special relativity

Lorentz Transformation of Space-Time

$$\begin{cases} t' &= \gamma\left(t - vx/c^2\right) \\ x' &= \gamma\left(x - vt\right) \\ y' &= y \\ z' &= z \end{cases}$$

O': Pure spatial seperation → O: Space-time mixed

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Lorentz Transformation of Acceleration (Force)

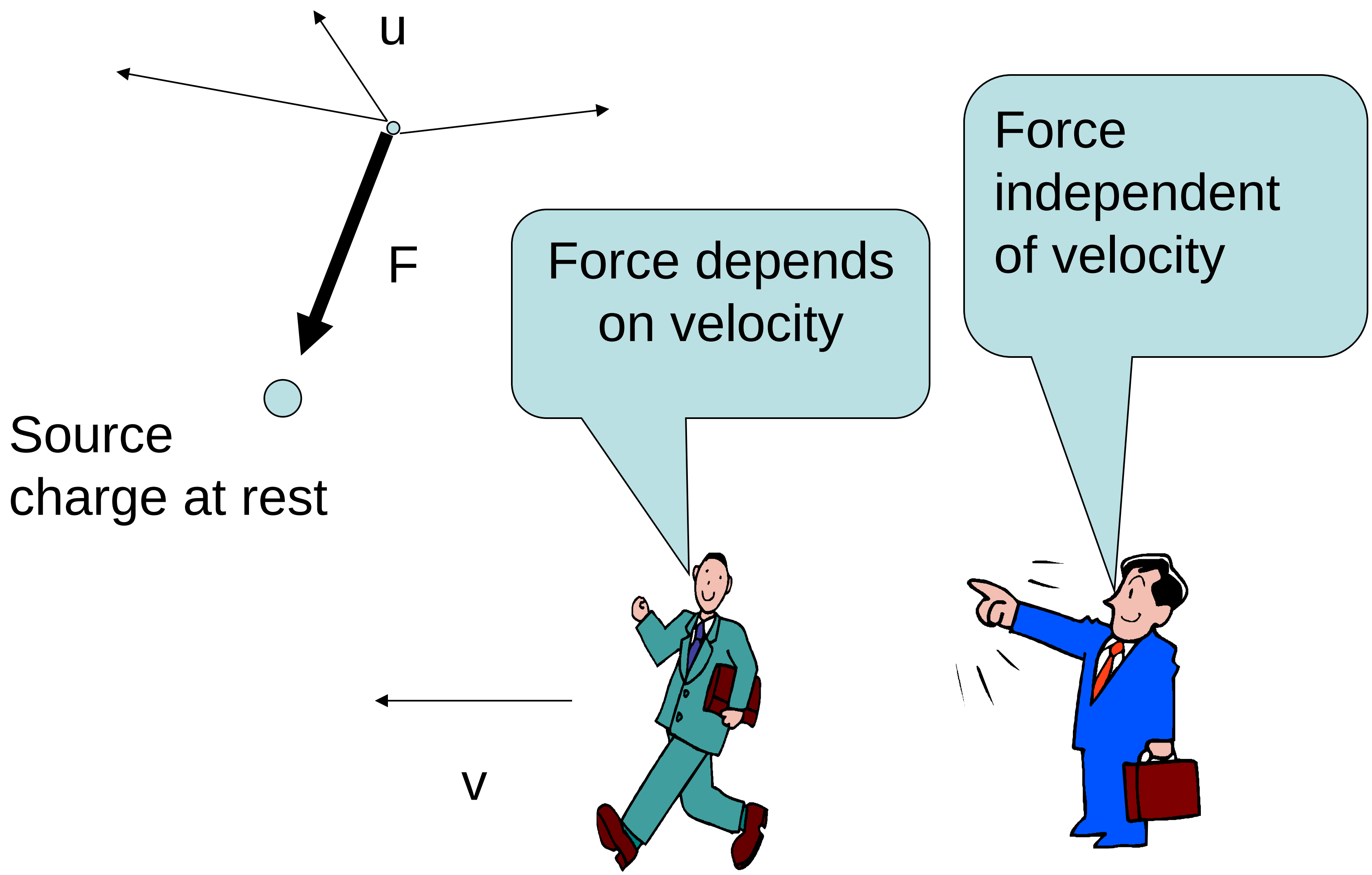
E.g., if a is independent of velocity for O

$$a'_x = \frac{a_x}{\gamma^3\left(1 - vu_x/c^2\right)^3}$$

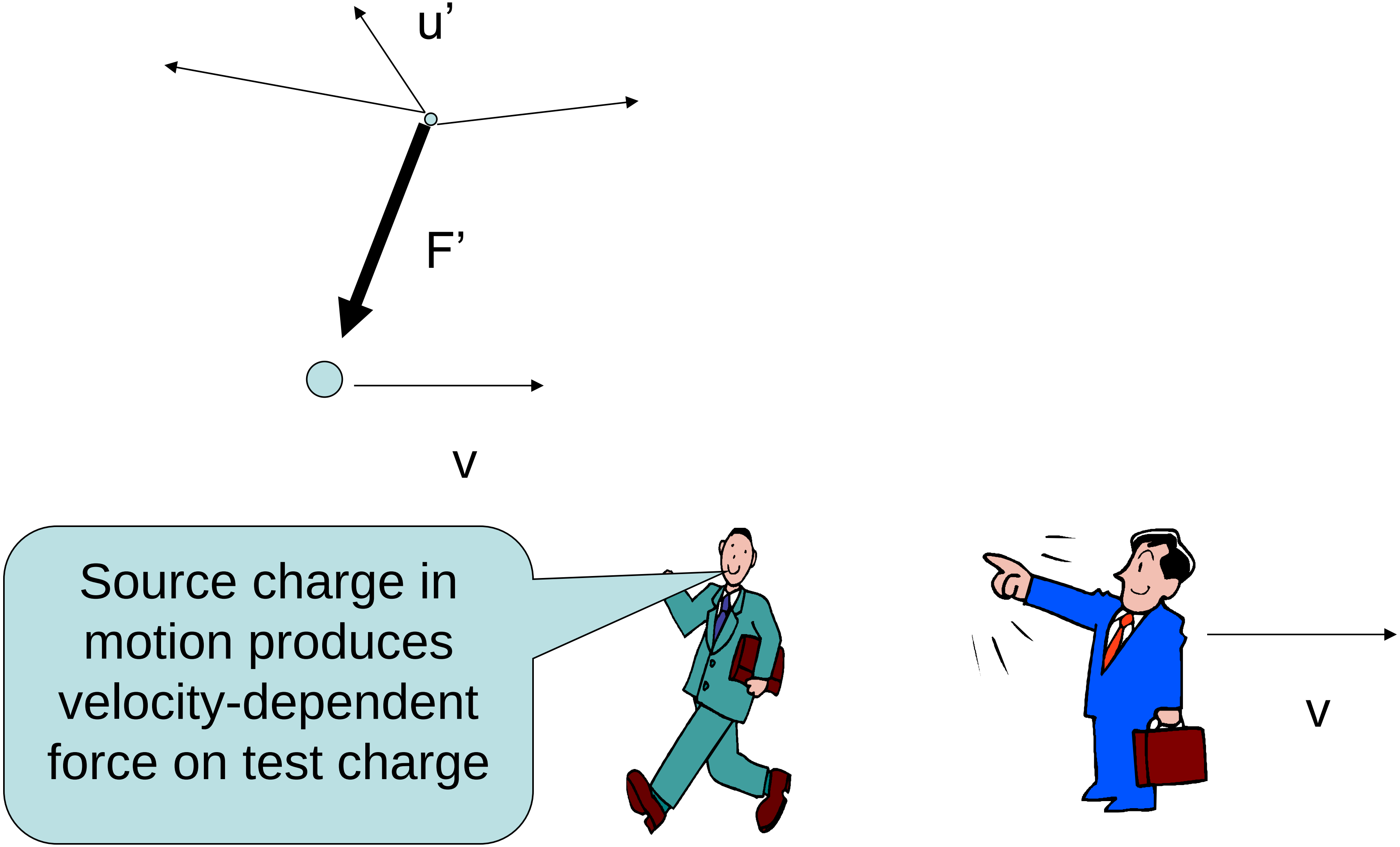
$$a'_y = \frac{(1 - vu_x/c^2)a_y + vu_y a_x/c^2}{\gamma^2\left(1 - vu_x/c^2\right)^3}$$

$$a'_z = \frac{(1 - vu_x/c^2)a_z + vu_z a_x/c^2}{\gamma^2\left(1 - vu_x/c^2\right)^3}$$

It becomes velocity dependent for O’

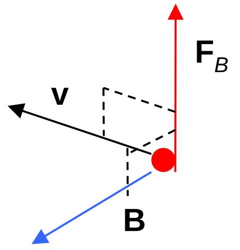

u
F
Source
charge at rest
v
Force depends
on velocity
Force
independent
of velocity

Transform to the frame of the second observer:



What if the source charges are moving?

- When the source charges are moving, it is found that there may be another force in addition to the electric force
- It is verified by experiments that this additional force is velocity-dependent and can be described by associating to every point in space a vector called the magnetic field **B**
- This force is then given by



$$\mathbf{F}_B = Q\mathbf{v} \times \mathbf{B}$$

- This is called the magnetic force

What if the source charges are moving?

Lorentz Transformation of the electromagnetic field

$$\vec{E}' = \gamma\left(\vec{E} + \vec{v}\times\vec{B}\right) + (1-\gamma)\frac{\vec{E}\cdot\vec{v}}{v^2}\vec{v}$$
$$\vec{B}' = \gamma\left(\vec{B} - \frac{1}{c^2}\vec{v}\times\vec{E}\right) + (1-\gamma)\frac{\vec{B}\cdot\vec{v}}{v^2}\vec{v}$$

In the non-relativistic limit: $v \ll c$ and $\gamma \approx 1$.

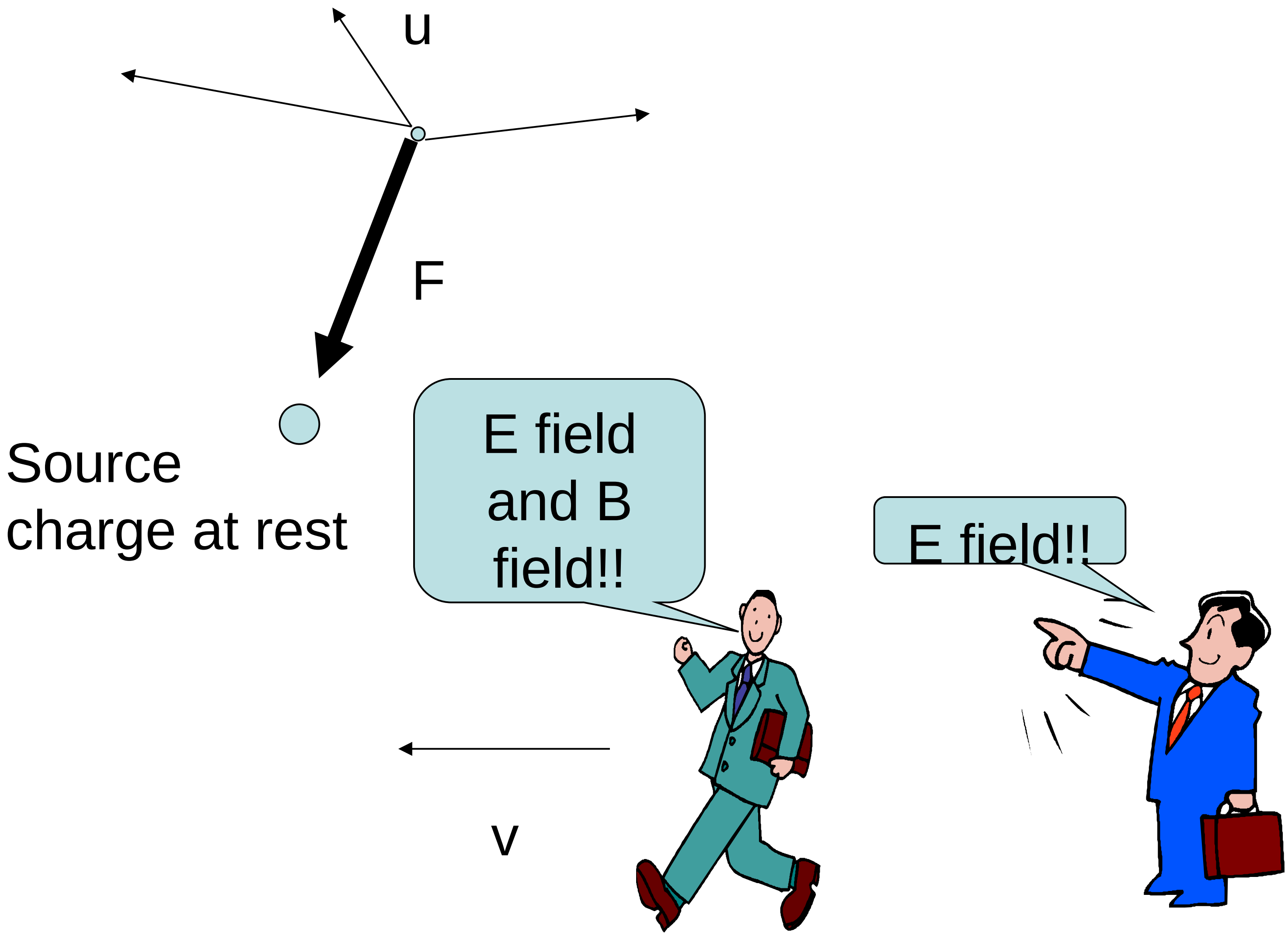
$$\vec{E}' = \vec{E} + \vec{v}\times\vec{B}$$
$$\vec{B}' = \vec{B} - (1/c^2)\vec{v}\times\vec{E} \approx \vec{B}$$

Lorentz Force Law

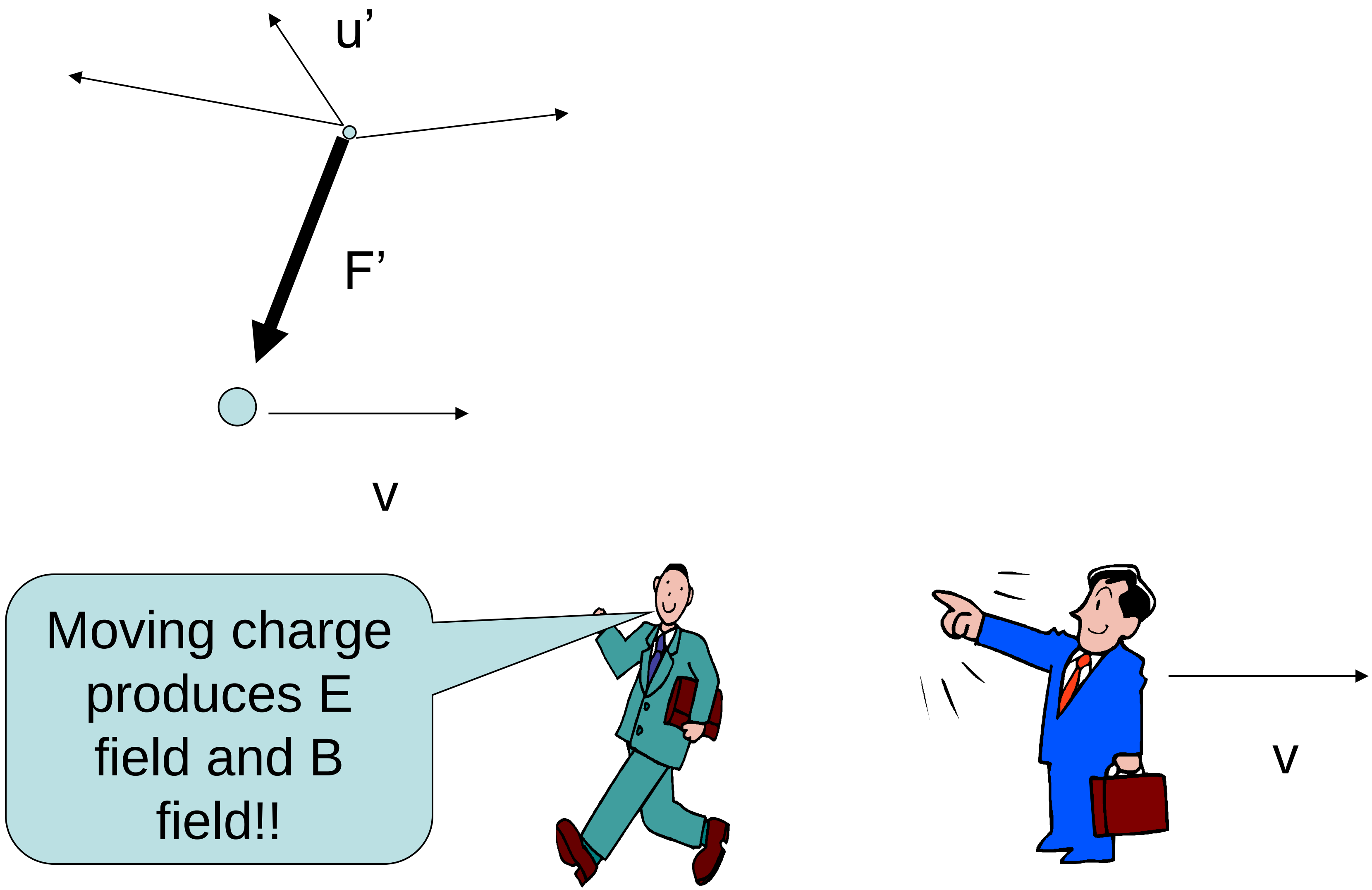
$$\mathbf{F} = Q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

Velocity-independent force
→ E field

Velocity-dependent force
→ B field


u
F
Source
charge at rest
E field
and B
field!!
E field!!
v

Transform to the frame of the second observer:



Magnetostatics

Application of Ampere’s Law

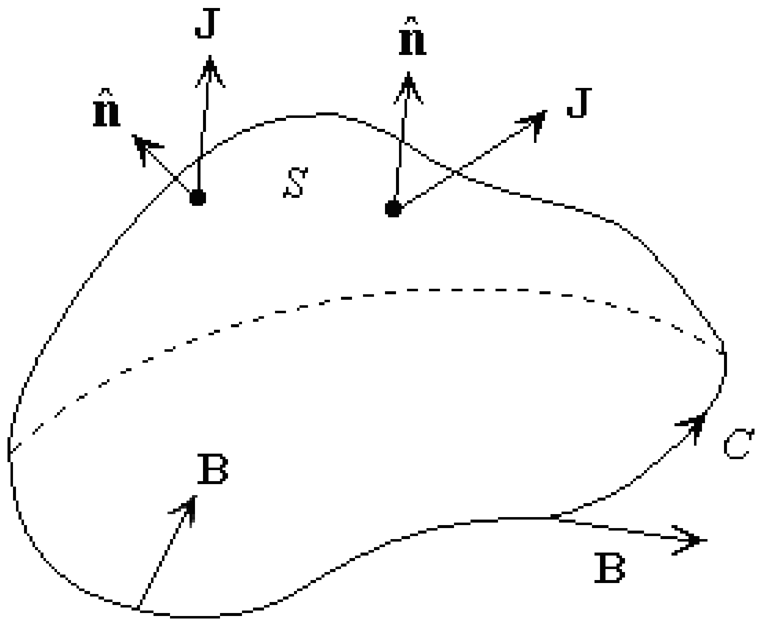
Integral Form of Ampere's Law

Consider a surface *S* with *C* as the boundary.

Stokes' thm:

$$\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \oint_C \mathbf{B} \cdot d\mathbf{l}$$

$$\therefore \quad \oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{a}$$



$I_{\text{enc}} = \int_S \mathbf{J} \cdot d\mathbf{a}$ is the amount of current enclosed by *C*

$$\therefore \quad \oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$$

Ampere's law in integral form

Current enclosed by loop C

$$I_{\text{enc}} = \int_S \mathbf{J} \cdot d\mathbf{a}$$

Is it well defined??

How to define “current passing through a loop”?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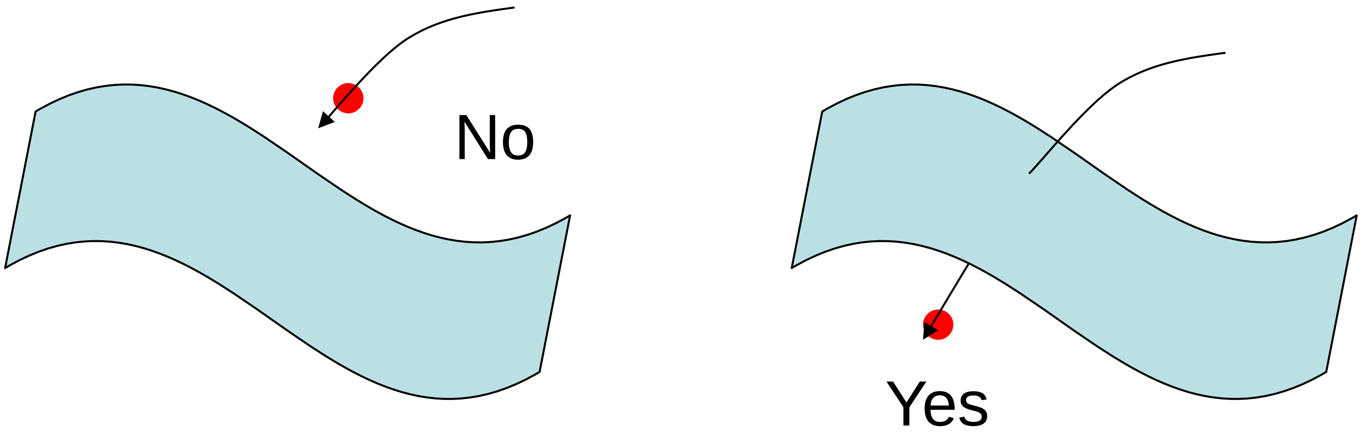
GOAL!!

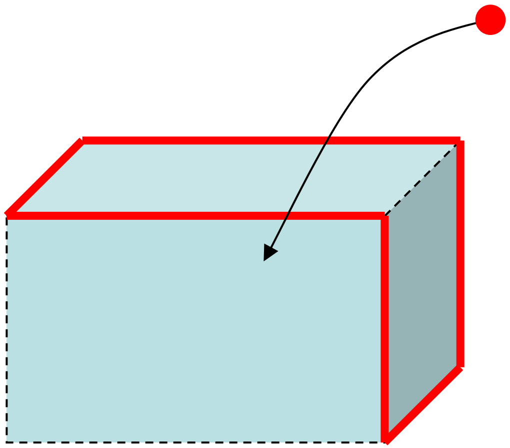
OR NOT?

- Note that one can only say whether something has passed through a surface



- "Passing through" a loop is not well-defined in general

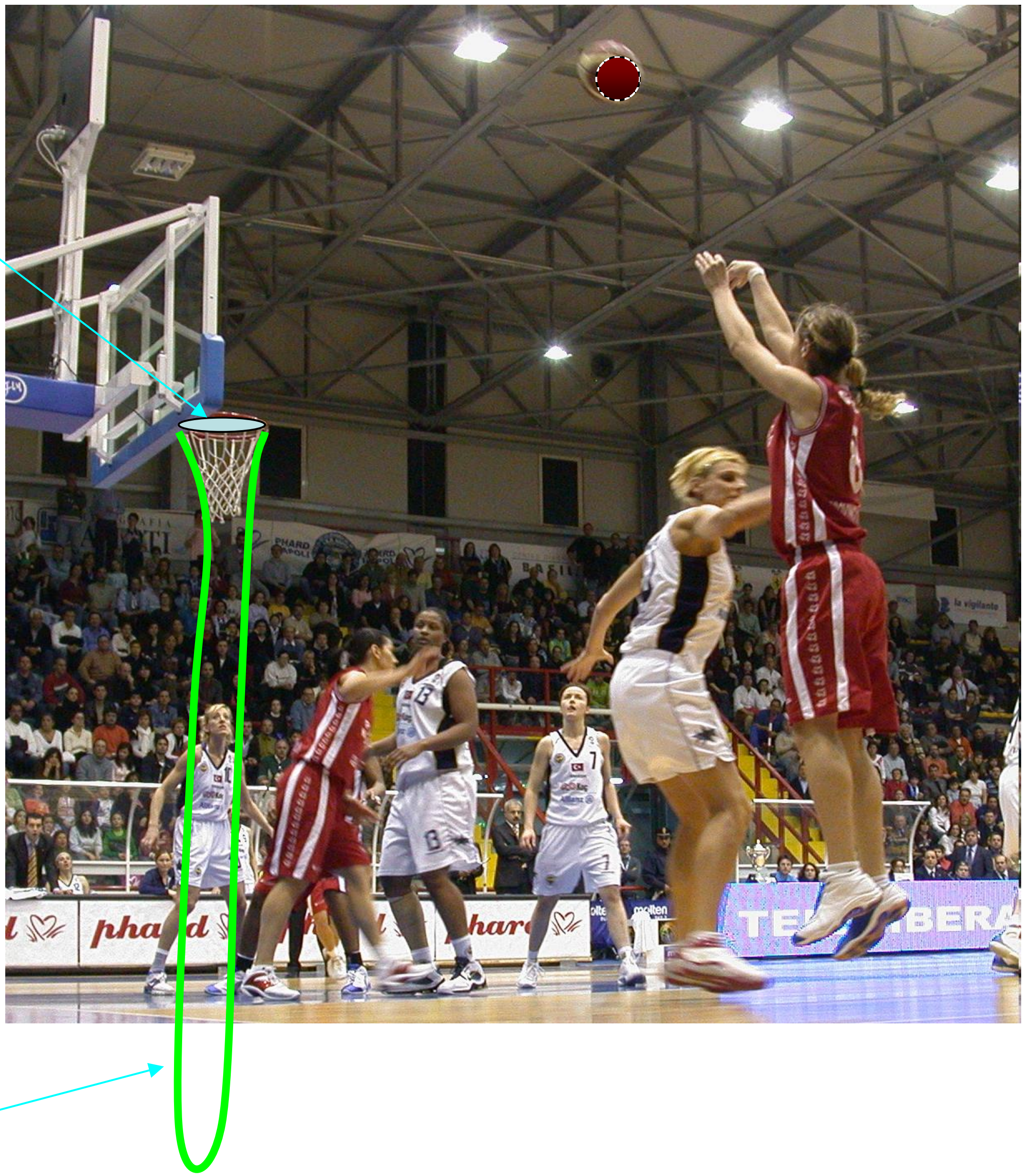
Yes or No?



- In our derivation of the integral form of Ampere's law, we chose an arbitrary surface S with the loop C as the boundary to define it
- But there is a problem!!

S_1

YES!!

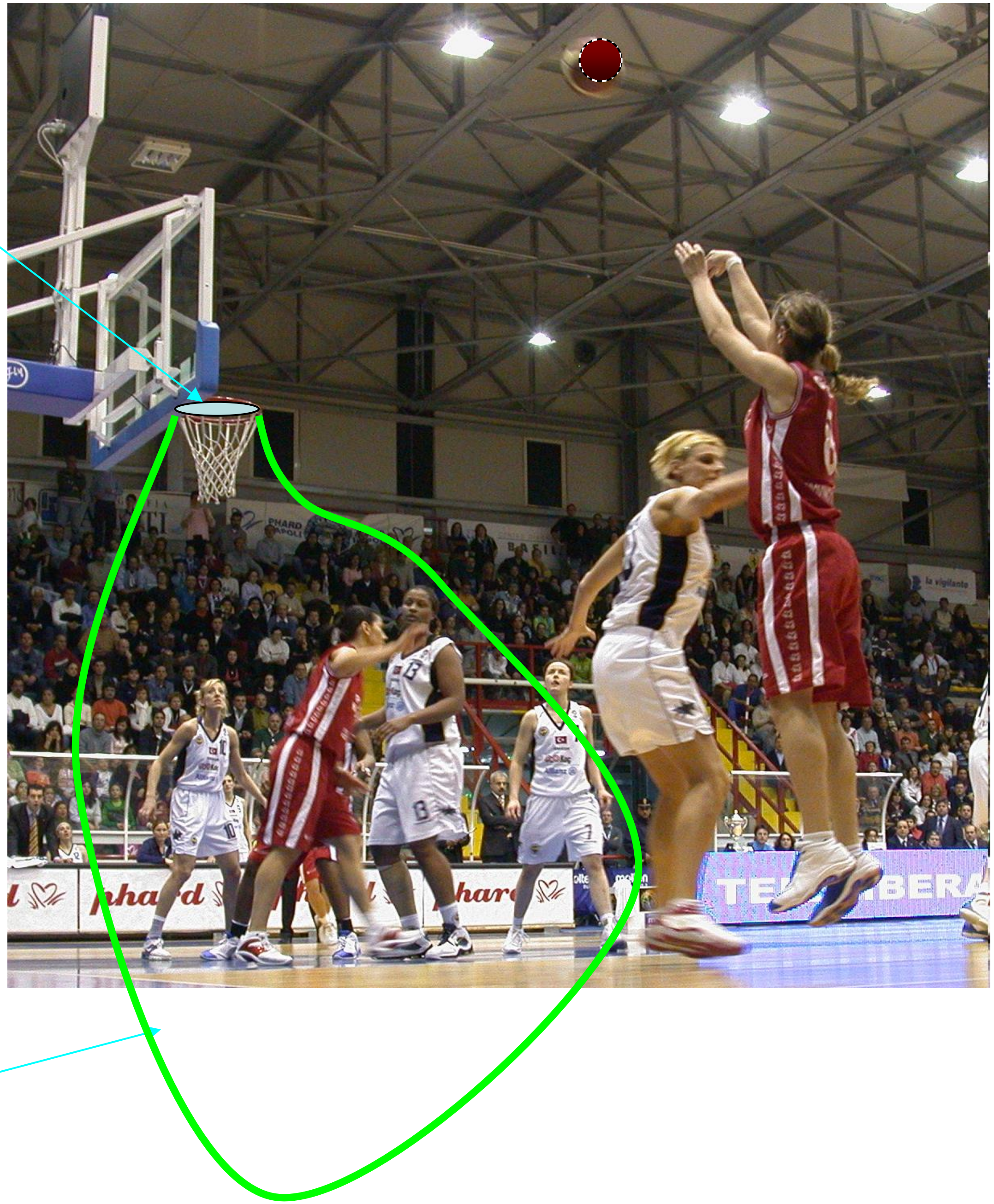


S_2

No!!

S_1

No!!

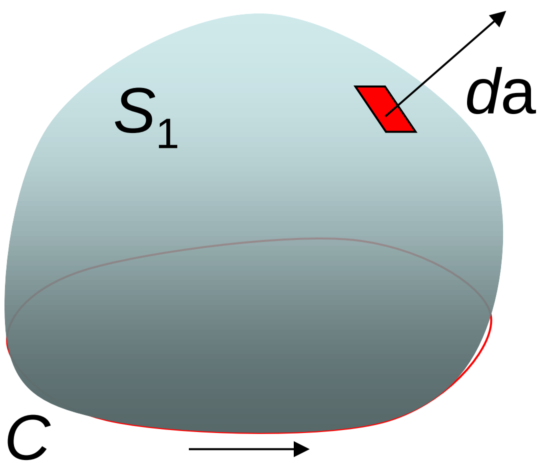


S_2

Yes!!

$$I_{\text{enc}} = \int_S \mathbf{J} \cdot d\mathbf{a}$$

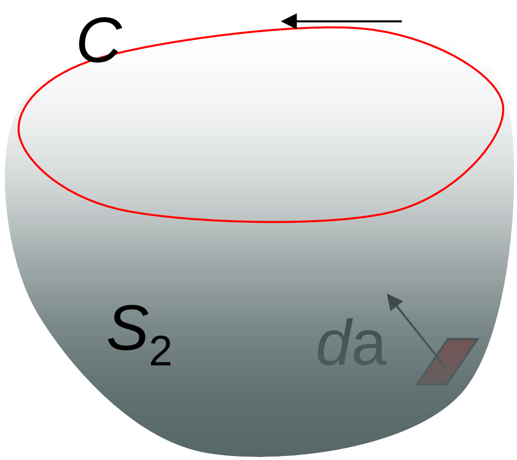
The current enclosed by a loop *C* is well-defined only when the surface integral is the same for all surface *S* with *C* as the boundary



$$\int_{S_1} \mathbf{J} \cdot d\mathbf{a}$$

$$=$$

$$\int_{S_2} \mathbf{J} \cdot d\mathbf{a}$$



Is this true?

Yes!

But only in the static regime!!

- In magnetostatics, the continuity equation implies

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = 0$$

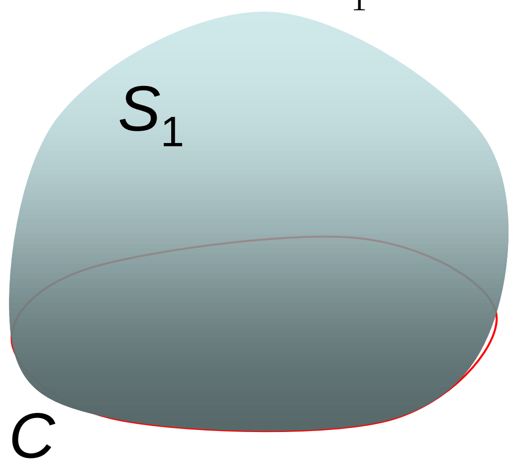
- By divergence theorem:

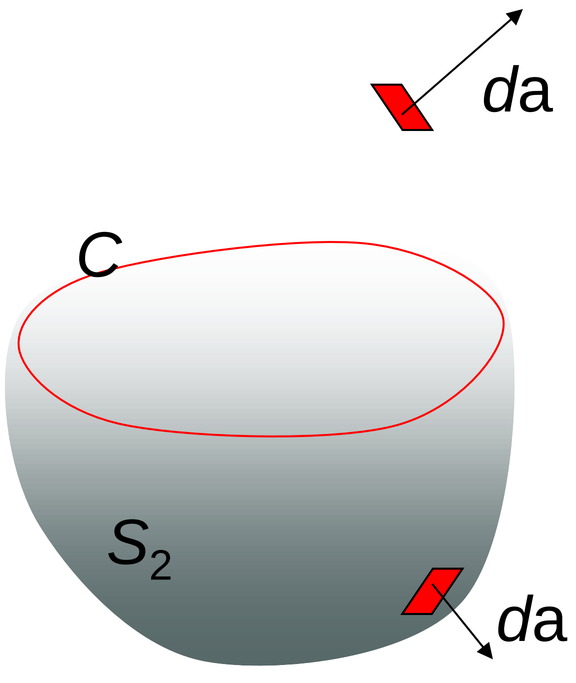
$$\oiint_{\Sigma} \mathbf{J} \cdot d\mathbf{a} = 0$$

for any arbitrary closed surface Σ

- For any two surfaces S_1 and S_2, one can combine them to form a closed surface Σ
- The total outward flux through Σ is zero
- The total flux consists of the flux through the two surfaces

$$\oint_{\Sigma} \mathbf{J}\cdot d\mathbf{a} = \int_{S_1} \mathbf{J}\cdot d\mathbf{a} + \int_{S_2} \mathbf{J}\cdot d\mathbf{a} = 0$$

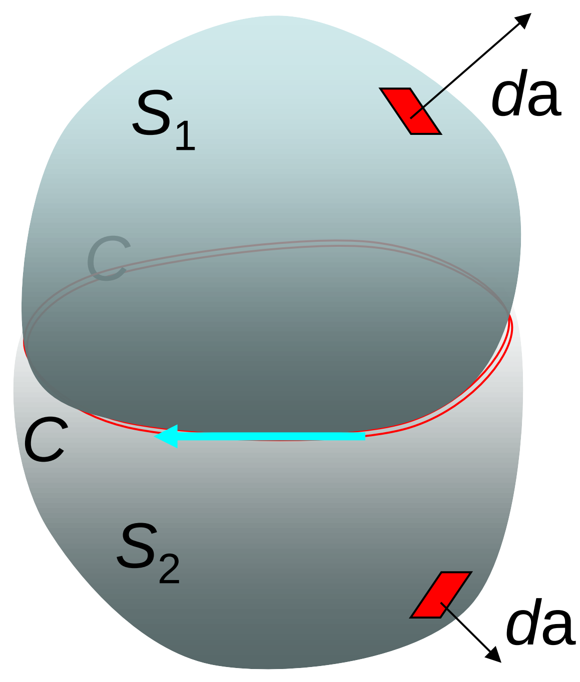




- If the flux of S_1 corresponds to the line integral on C along a certain direction
- then, the flux of S_2 corresponds to the line integral in the opposite direction

$$\oint_C \mathbf{B}\cdot d\mathbf{l} = \int_{S_1} \mathbf{J}\cdot d\mathbf{a}$$

$$\oint_C \mathbf{B}\cdot d\mathbf{l} = \int_{S_2} \mathbf{J}\cdot d\mathbf{a}$$



$$\int_{S_1} \mathbf{J} \cdot d\mathbf{a} + \int_{S_2} \mathbf{J} \cdot d\mathbf{a} = 0 \Rightarrow$$

$$\int_{S_1} \mathbf{J} \cdot d\mathbf{a} = -\int_{S_2} \mathbf{J} \cdot d\mathbf{a}$$

- Change the direction of $d\mathbf{a}$ of S_2 lead to the same direction of line integral
- Hence

$$\Rightarrow \int_{S_1} \mathbf{J} \cdot d\mathbf{a} = \int_{S_2} \mathbf{J} \cdot d\mathbf{a}$$

$$\int_S \mathbf{J} \cdot d\mathbf{a}$$

is the same for all surface *S* with *C* as the boundary

$$I_{\text{enc}} = \int_S \mathbf{J} \cdot d\mathbf{a}$$

is well defined