Chapter 5

Magnetostatics

History

- At first, Electricity and Magnetism appeared to the separated, unrelated subjects
- Electricity deals with forces between charges
- Magnetism deals with forces between magnets

The year 1820

- July 21, **Hans Christian Oersted** noted the deflection of a magnetic compass needle caused by an electric current.
- July 27, **André Marie Ampère** confirmed Oersted's results and presented extensive experimental results to the French Academy of Science.
- He modeled magnets in terms of molecular electric currents.
- He discovered electrodynamical forces between linear wires before the end of September.
- Initiated the unification program of electricity and magnetism.

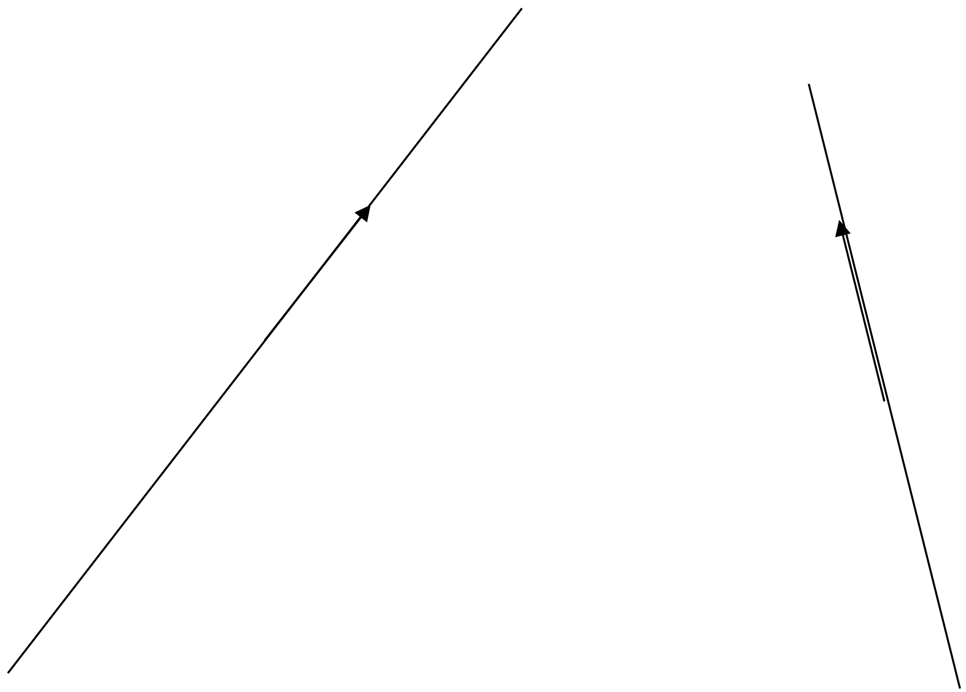
Hans Christian Ørsted
(1777-1851)

André Marie Ampère
(1775 - 1836)

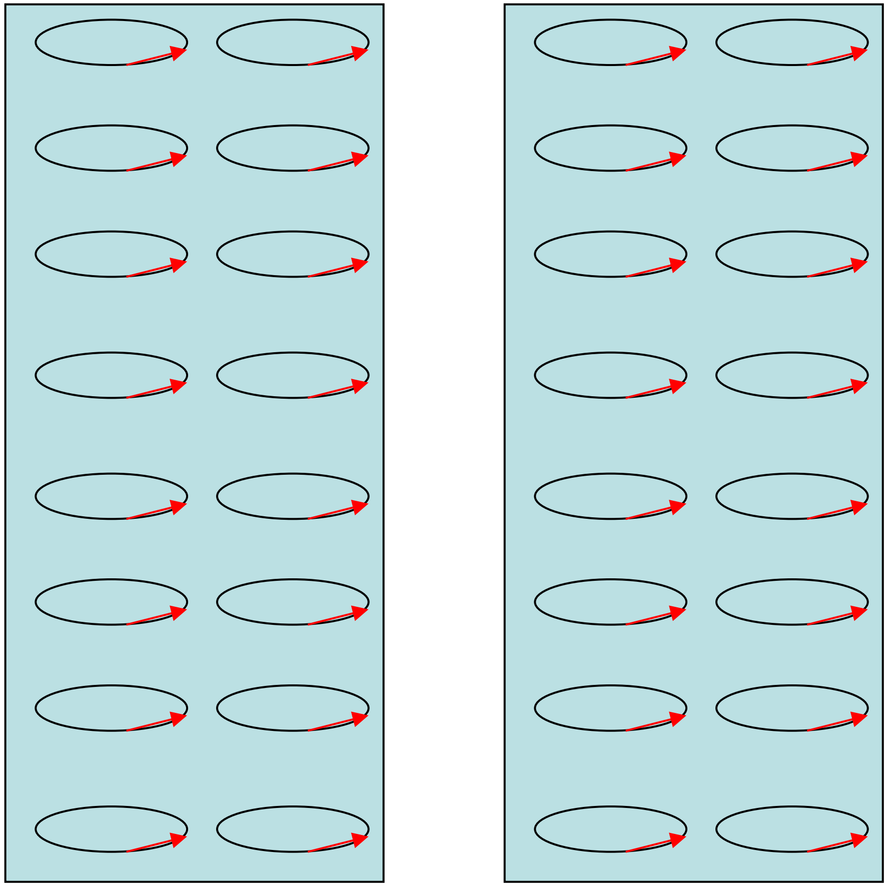
Forces Between Wires

Force is observed between two wires carrying currents

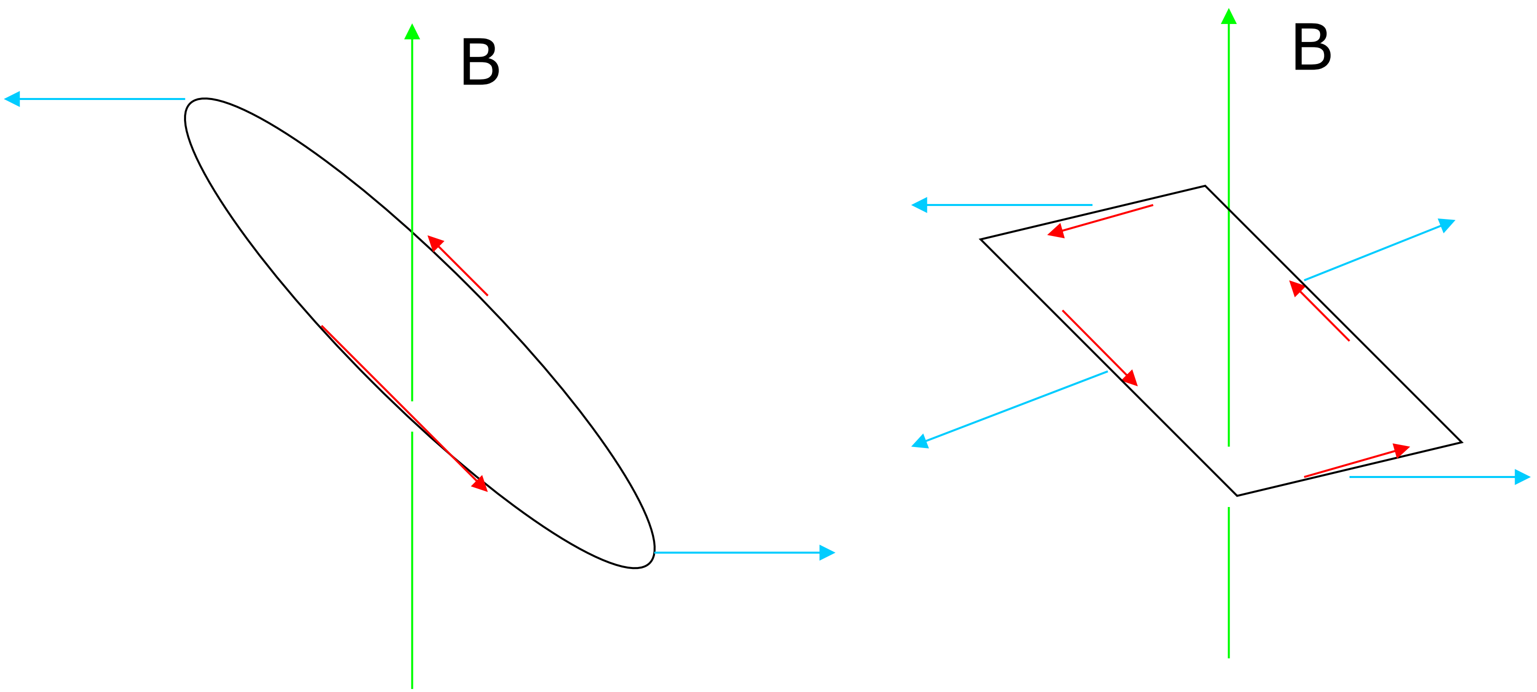
Note: A test charge at rest near the wires experiences no force



Forces Between Magnets



Forces Between Magnets

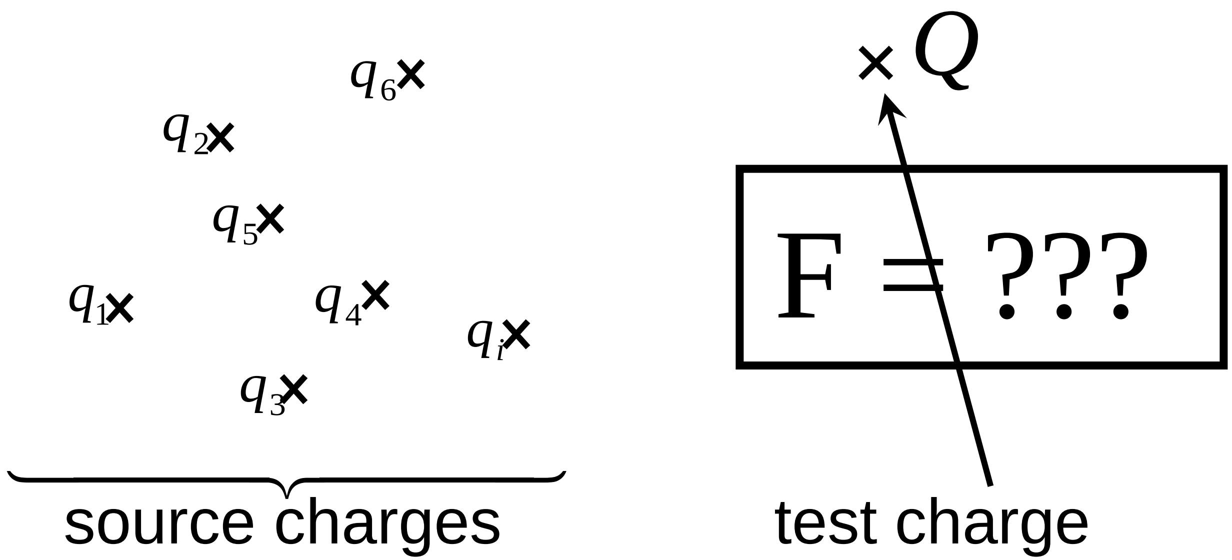


Currents (Charges in motion) Produce Magnetic Fields

Why???

- Electrostatics: Source charges **at rest**
- Magnetostatics: Source charges moving, giving rise to steady currents and constant current densities

What is the force exerted on a test charge Q, by some source charges $q_1, q_2, q_3, \cdots$?



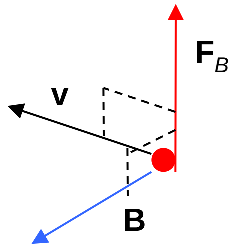
- When the source charges are at rest, it is observed that the force acting on the test charge is in general position dependent but independent of the motion of the test charge
- Hence one can
  - assigning to the test charge a number Q, called its charge
  - assigning to every point in space a vector called the electric field **E**
- The force can then be given by

$$\mathbf{F}_E = Q\mathbf{E}$$

- This is called the electric force

What if the source charges are moving?

- When the source charges are moving, it is found that there may be another force in addition to the electric force
- It is verified by experiments that this additional force is velocity-dependent and can be described by associating to every point in space a vector called the magnetic field **B**
- This force is then given by



$$\mathbf{F}_B = Q\mathbf{v} \times \mathbf{B}$$

- This is called the magnetic force

Lorentz Force Law

$$\mathbf{F} = Q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

Velocity-independent force
→ E field

Velocity-dependent force
→ B field

Magnetic forces do no work

If the charge moves a displacement

$$d\mathbf{l} = \mathbf{v}dt$$

The work done by the magnetic force $\mathbf{F_B}$ is

$$dW_B = \mathbf{F_B} \cdot d\mathbf{l} = Q\left(\mathbf{v} \times \mathbf{B}\right) \cdot \mathbf{v}dt$$

But

$$\mathbf{v} \times \mathbf{B} \perp \mathbf{v}$$

$$\therefore dW_B = 0$$

Example: Cyclotron motion

Consider a charge Q in a uniform magnetic field $\mathbf{B}$. The velocity $\mathbf{v}$ of the charge is perpendicular to $\mathbf{B}$.

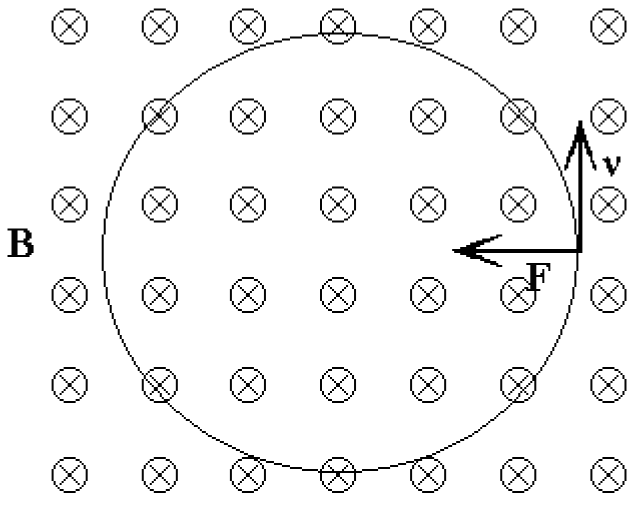
By Lorentz force law,

$$F = QvB = m\frac{v^2}{R}$$

where $\boldsymbol{R}$ is the radius of the circle

$$\therefore p = mv = QBR$$

(cyclotron formula)

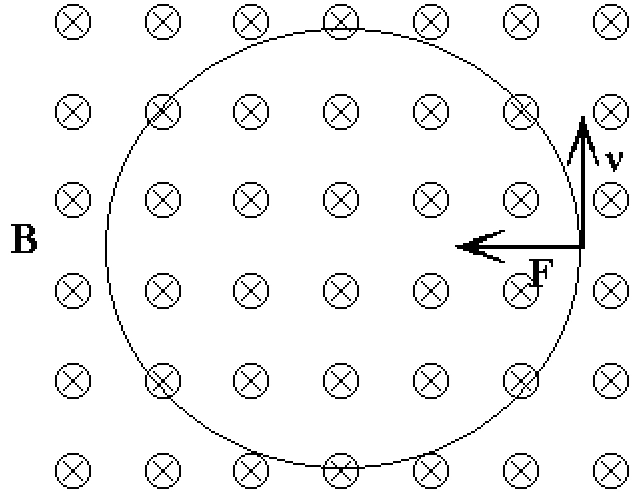


Example: Cyclotron motion

If **v** has a component parallel to **B**:

$$\mathbf{v} = \mathbf{v}^{//} + \mathbf{v}^{\perp}$$

$$\begin{aligned}\mathbf{F} &= Q\mathbf{v} \times \mathbf{B} \\ &= Q\left(\mathbf{v}^{//} + \mathbf{v}^{\perp}\right) \times \mathbf{B} \\ &= Q\mathbf{v}^{\perp} \times \mathbf{B}\end{aligned}$$



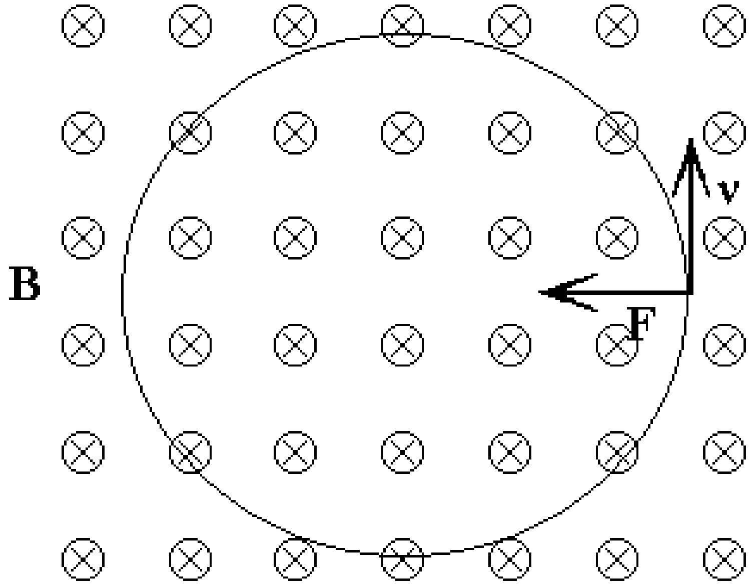
Besides, $\mathbf{F} \perp \mathbf{B}$

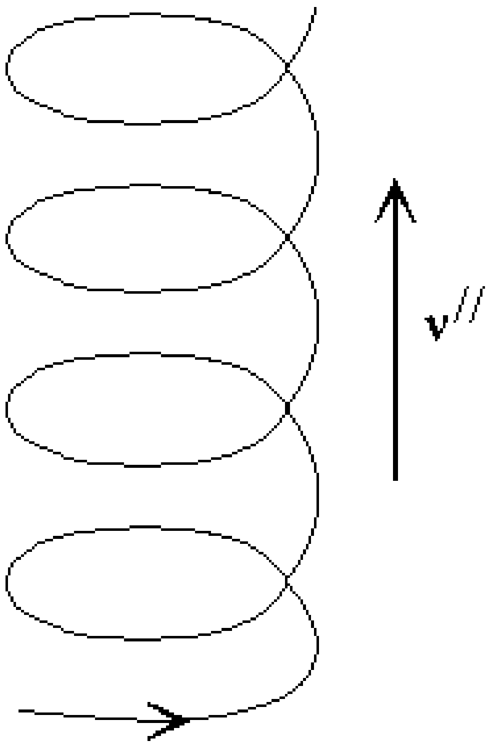
$$\therefore \mathbf{F} \perp \mathbf{v}^{//}$$

Therefore, $\mathbf{v}^{//}$ is unchanged.

Example: Cyclotron motion

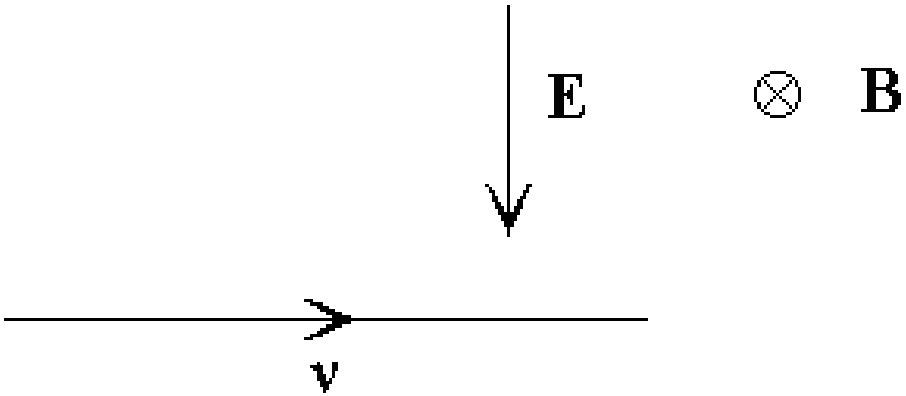
The particle moves in a helix.





Example: Electron charge-mass ratio

Consider an electron moving in a region of uniform E-field and B-field.



If the fields are adjusted such that the electron experiences no net force and moves with a constant velocity **v**

Then

$$eE = evB$$

$$\therefore v = E/B$$

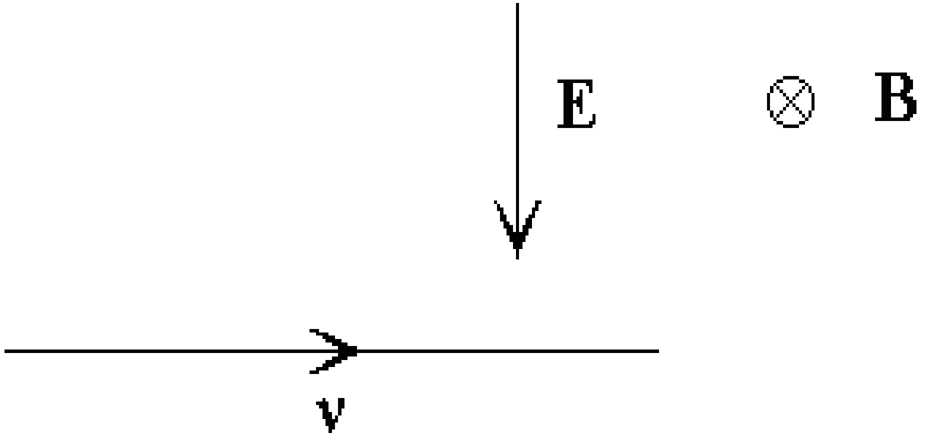
Example: Electron charge-mass ratio

Switch off the E-field and measure the radius of the circular trajectory, ***R***,

$$mv = eBR$$

$$\therefore \frac{eBR}{m} = \frac{E}{B}$$

$$\frac{e}{m} = \frac{E}{B^2 R}$$



Currents

- Currents are due to the motion of charges
- It measures the rate of flow of charges
- the SI unit of current : ampere (A)
- One ampere means there is one Coulomb of charges flowing through in one second

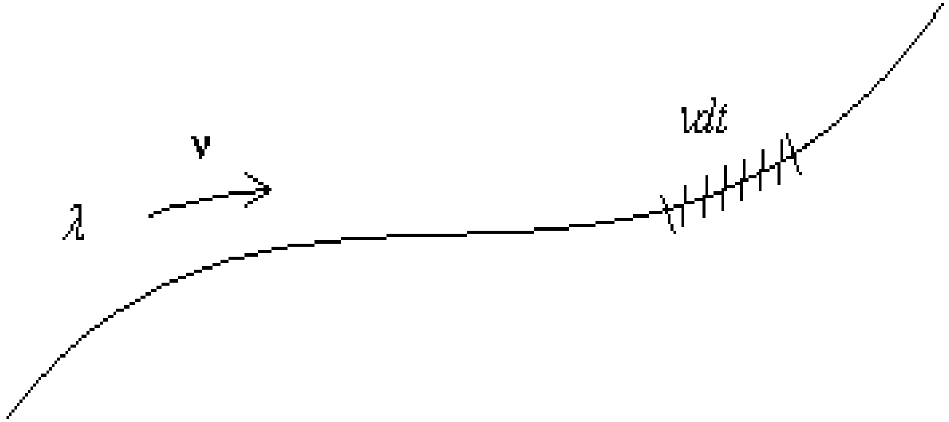
$$\therefore 1\mathrm{A} = 1\mathrm{C/s}$$

Currents

- **Current I = rate of flux of charges**
- Current has both magnitude and direction
- It is a vector
- Magnitude: $I = dq/dt$
- Direction is determined by the motion of charges
  - In most situations, it is due to the flow of negative charges (electrons) in a certain direction
  - But conventionally, we imagine that it is due to the flow of positive charges in the opposite direction
- Direction of current:
  - The same direction of the flow of positive charges
  - Opposite direction to the flow of negative charges

Line Current

- Charges flowing along a "wire" with negligible cross section area.
- Linear charge density λ
- Charges inside moving at velocity **v**



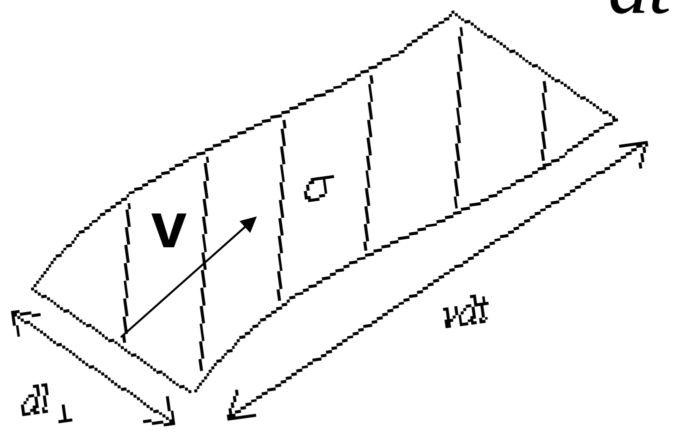
$$dq = \lambda v dt$$

$$\therefore \mathbf{I} = \lambda \mathbf{v}$$

Surface Current Density

- Charges flowing inside a "sheet" with negligible thickness
- Surface charge density σ
- Charges moving with velocity **v**

Then $dq = \sigma dl_{\perp} v dt \quad \rightarrow \quad \dfrac{dq}{dt} = \sigma v dl_{\perp}$

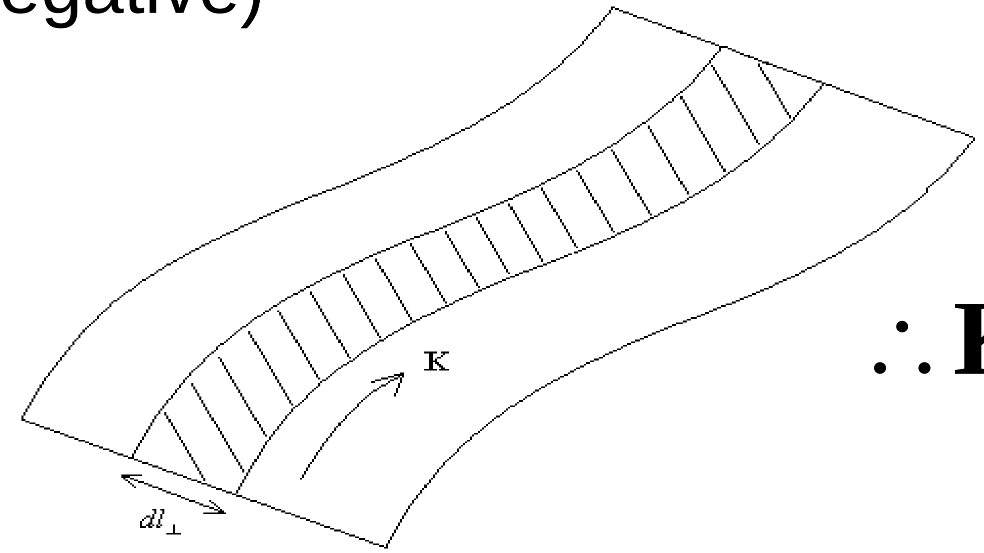


Surface Current Density

- Def: Surface current density **K**:
  - Magnitude: Rate of charge flow per unit length-perpendicular-to-flow

$$K = \frac{1}{dl_\perp}\frac{dq}{dt} = \sigma v$$

  - Direction: **v** (if σ is positive), -**v** (if σ is negative)



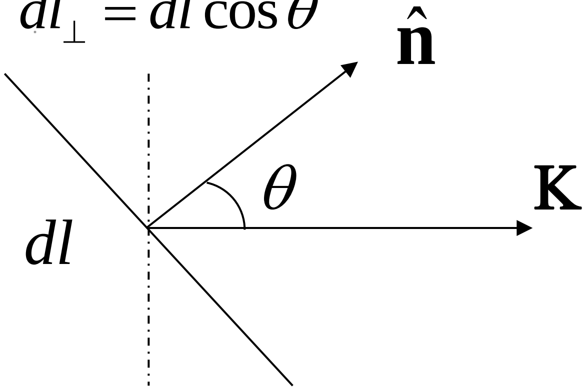
$$\therefore \mathbf{K} = \sigma \mathbf{v}$$

Surface Current Density

- In general, if the unit vector $\hat{\mathbf{n}}$, which is perpendicular to the line segment, makes an angle θ with the direction of **K**, the rate of flow in the direction of $\hat{\mathbf{n}}$ is

$$\frac{dq}{dt} = \mathbf{K} \cdot dl\hat{\mathbf{n}}$$

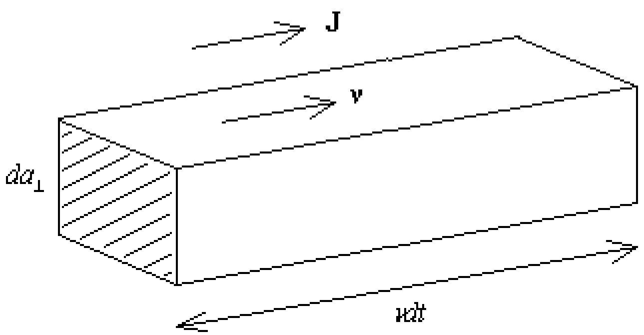
$$dl_{\perp} = dl \cos\theta$$



Volume Current Density

- Charges flowing inside a volume
- The volume charge distribution ρ.
- The charges are moving with velocity **v**.

Then $$dq = \rho v dt \cdot da_{\perp} \quad \rightarrow \quad \frac{dq}{dt} = \rho v da_{\perp}$$

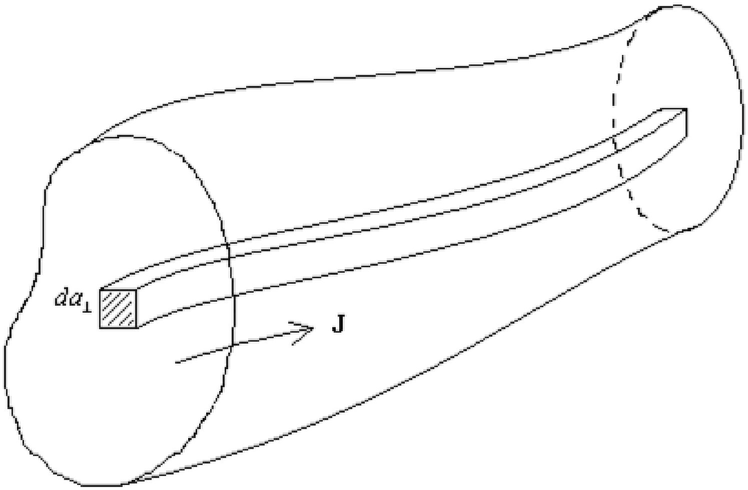


Volume Current Density

- Def: Volume current density **J**:
  - Magnitude: Rate of charge flow per unit area-perpendicular-to-flow

$$J = \frac{1}{da_\perp}\frac{dq}{dt} = \rho v$$

  - Direction: **v** (if ρ is positive), -**v** (if ρ is negative)

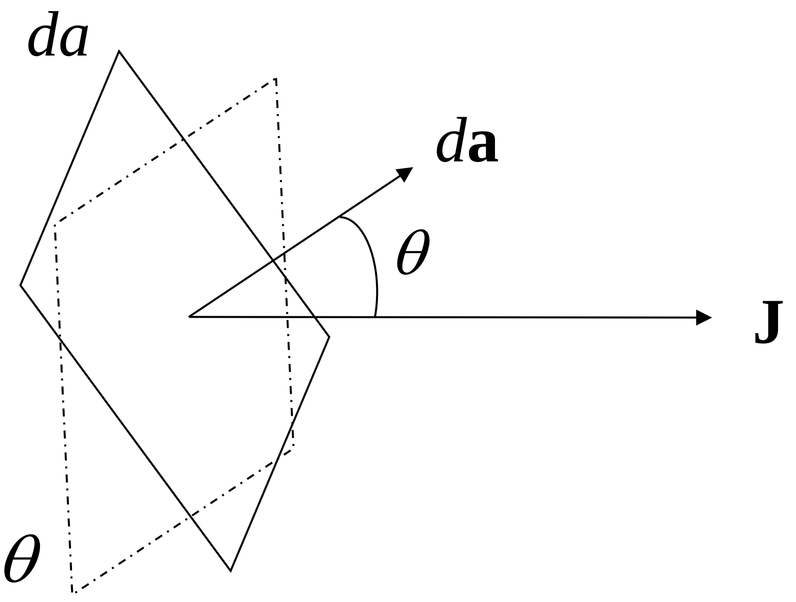


$$\therefore \mathbf{J} = \rho \mathbf{v}$$

Volume Current Density

- In general, if the area element $d\mathbf{a}$ makes an angle θ with the direction of **J**, the flux in the direction of $d\mathbf{a}$ is

$$\frac{dq}{dt} = \mathbf{J} \cdot d\mathbf{a}$$



$$da_{\perp} = da \cos\theta$$

Continuity Equation

(Derivation in 3D here. The derivations in 1D and 2D are similar.)

Consider the current crossing a closed surface S:

$$I = \oint_S J da_\perp = \oint_S \mathbf{J} \cdot d\mathbf{a}$$

$$= \int_{\mathcal{V}} (\nabla \cdot \mathbf{J}) \, d\tau$$

Since charge is conserved locally,

$$\frac{d}{dt} \int_{\mathcal{V}} \rho d\tau = -\oint_S \mathbf{J} \cdot d\mathbf{a}$$

$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} d\tau = -\int_{\mathcal{V}} (\nabla \cdot \mathbf{J}) \, d\tau$$

Continuity Equation

(Derivation in 3D here. The derivations in 1D and 2D are similar.)

$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} d\tau = -\int_{\mathcal{V}} \nabla \cdot \mathbf{J} d\tau$$

Since this is true for arbitrary volume $\mathcal{V}$, hence

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0}$$ ⟵ ***continuity equation***

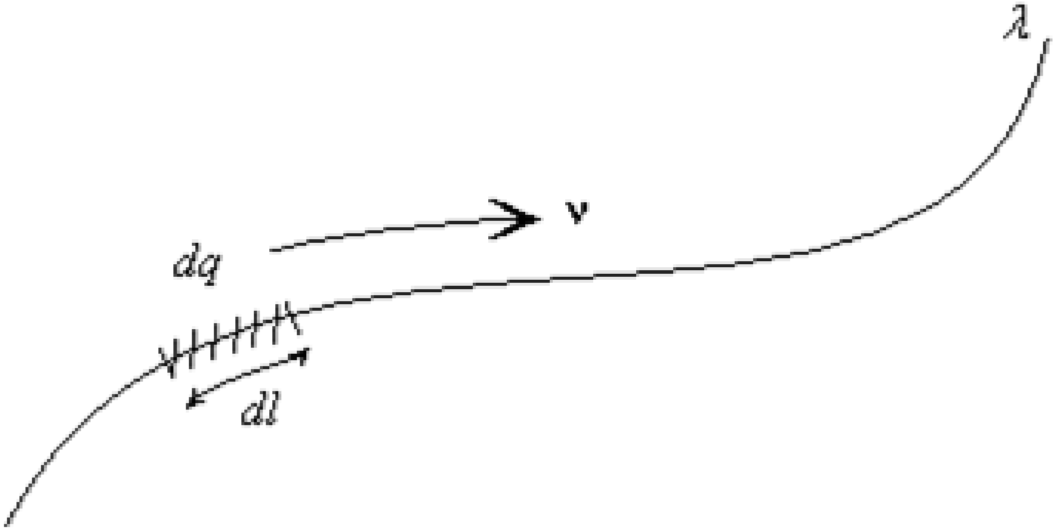
Force Experienced by Currents Inside Given B Field

- Current are due to charges in motion.
- By Lorentz force law, moving charges will experience magnetic forces in B field.
- Hence, inside B field, with no E field →

Magnetic Force on Currents

- *Line Current:*

$$\mathbf{F_B} = \int dq\left(\mathbf{v}\times\mathbf{B}\right) = \int\left(\mathbf{v}\times\mathbf{B}\right)\lambda dl = \int\left(\mathbf{I}\times\mathbf{B}\right)dl$$



Magnetic Force on Currents

- *Line Current:*

For line current, $\mathbf{I}$ is along the wire. So we define $d\mathbf{l}$ with the same direction as $\mathbf{I}$.

$$\therefore \mathbf{I}dl = Id\mathbf{l}$$

$$\mathbf{F_B} = \int (\mathbf{I} \times \mathbf{B}) dl$$

$$\downarrow$$

$$\therefore \mathbf{F_B} = \int I (d\mathbf{l} \times \mathbf{B})$$

Magnetic Force on Currents

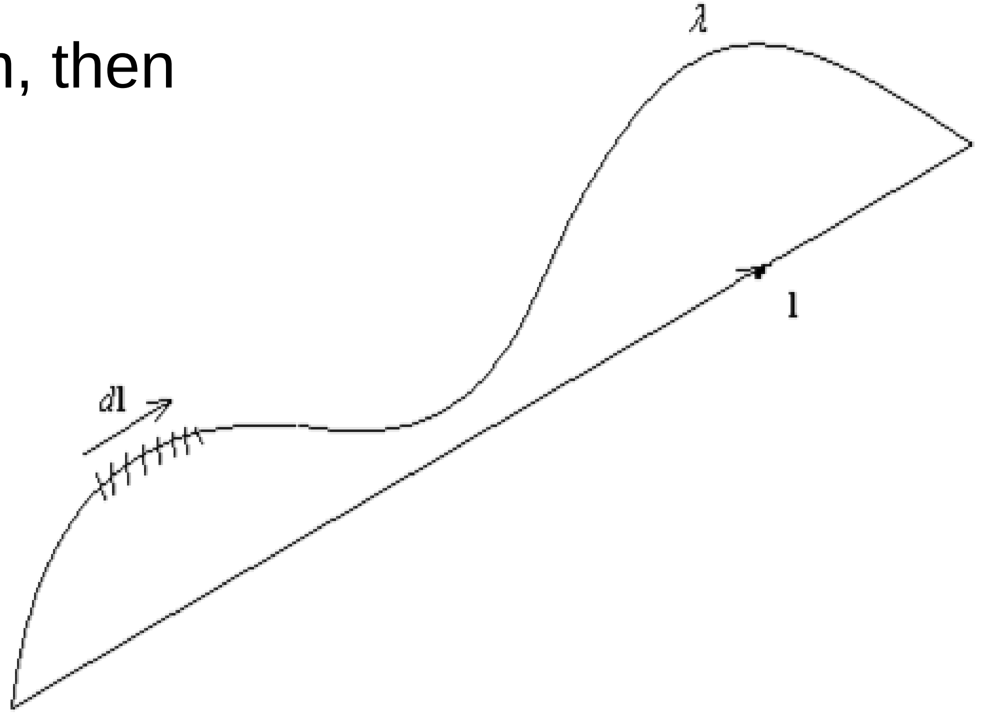
- *Line Current:*

$$\therefore \mathbf{F_B} = \int I(d\mathbf{l} \times \mathbf{B})$$

If the $\mathbf{B}$-field is uniform, then

$$\mathbf{F_B} = I\left(\int d\mathbf{l}\right) \times \mathbf{B}$$

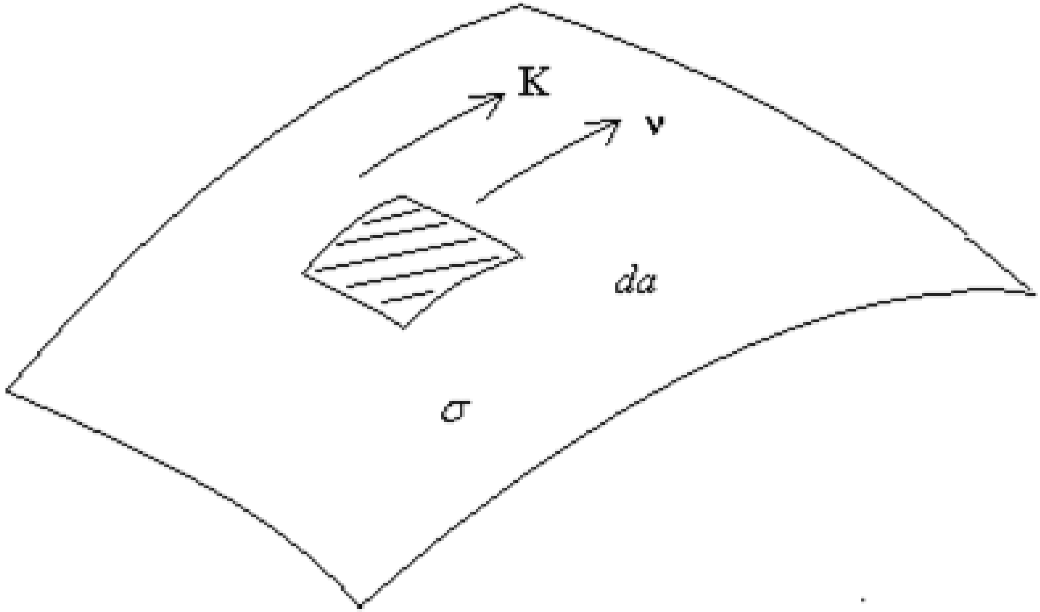
$$= I\mathbf{l} \times \mathbf{B}$$



Magnetic Force on Currents

- *Surface Current Density :*

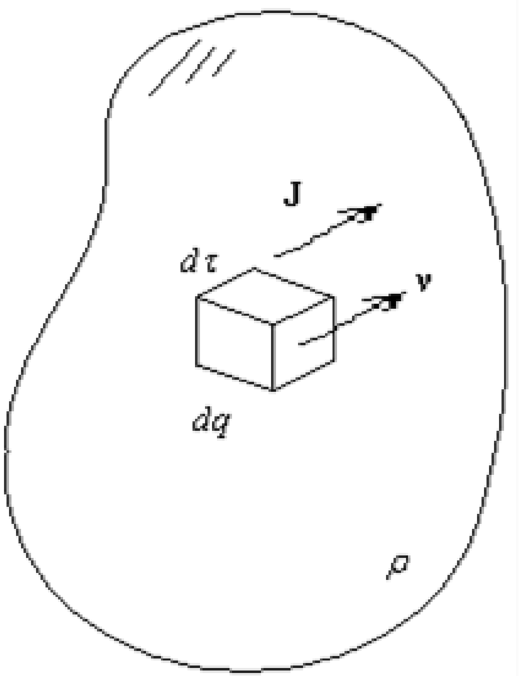
$$\mathbf{F_B} = \int dq(\mathbf{v}\times\mathbf{B}) = \int(\mathbf{v}\times\mathbf{B})\sigma da = \int(\mathbf{K}\times\mathbf{B})da$$



Magnetic Force on Currents

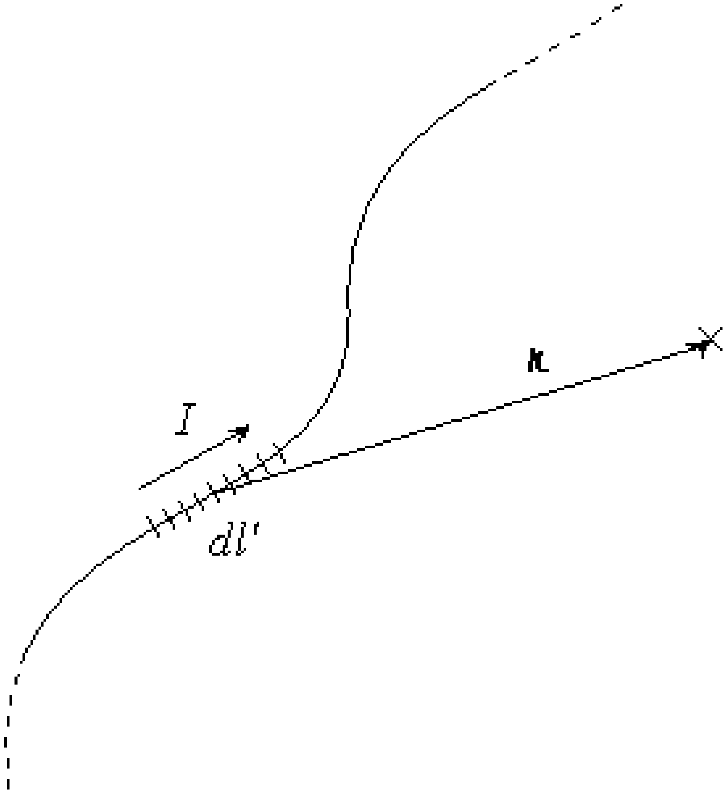
- *Volume Current Density :*

$$\mathbf{F_B} = \int dq(\mathbf{v}\times\mathbf{B}) = \int(\mathbf{v}\times\mathbf{B})\rho d\tau = \int(\mathbf{J}\times\mathbf{B})d\tau$$



Biot-Savart Law

The magnetic field of a steady line current is give by



$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}\times\hat{r}}{r^2}\,dl'$$

where $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{N/A^2}$ *(exactly!!)*

permeability of free space

Biot-Savart Law

For a line current,

I is along the direction of the wire

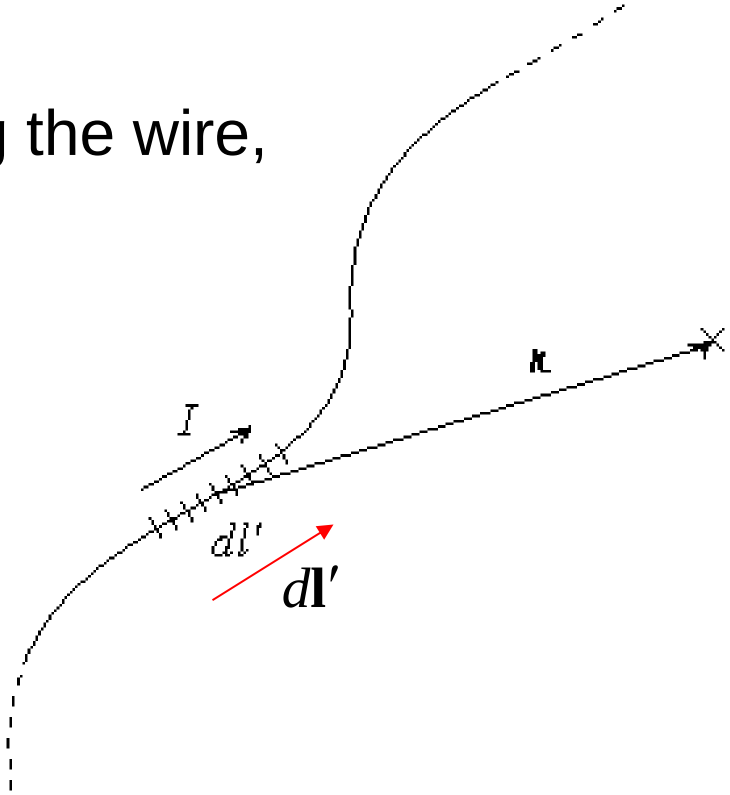
$$\therefore \mathbf{I}\,dl' = I\,d\mathbf{l}'$$

and usually I is constant along the wire,

so it can also be written as,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}\times\hat{\boldsymbol{\imath}}}{\imath^2}\,dl'$$

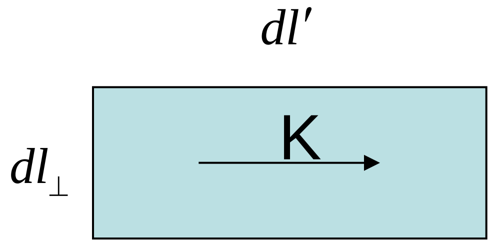
$$= \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}'\times\hat{\boldsymbol{\imath}}}{\imath^2}$$



Biot-Savart Law

For a line current $\mathbf{I}$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I} \times \hat{\mathscr{r}}}{\mathscr{r}^2} dl'$$



For a surface current density $\mathbf{K}(\mathbf{r}')$

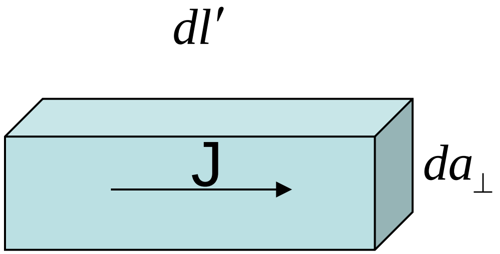
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K} \times \hat{\mathscr{r}}}{\mathscr{r}^2} da'$$

$$\mathbf{I} = \mathbf{K} dl_{\perp}$$

$$\mathbf{I} dl' = \mathbf{K} dl_{\perp} dl' = \mathbf{K} da'$$

For a volume current density $\mathbf{J}(\mathbf{r}')$

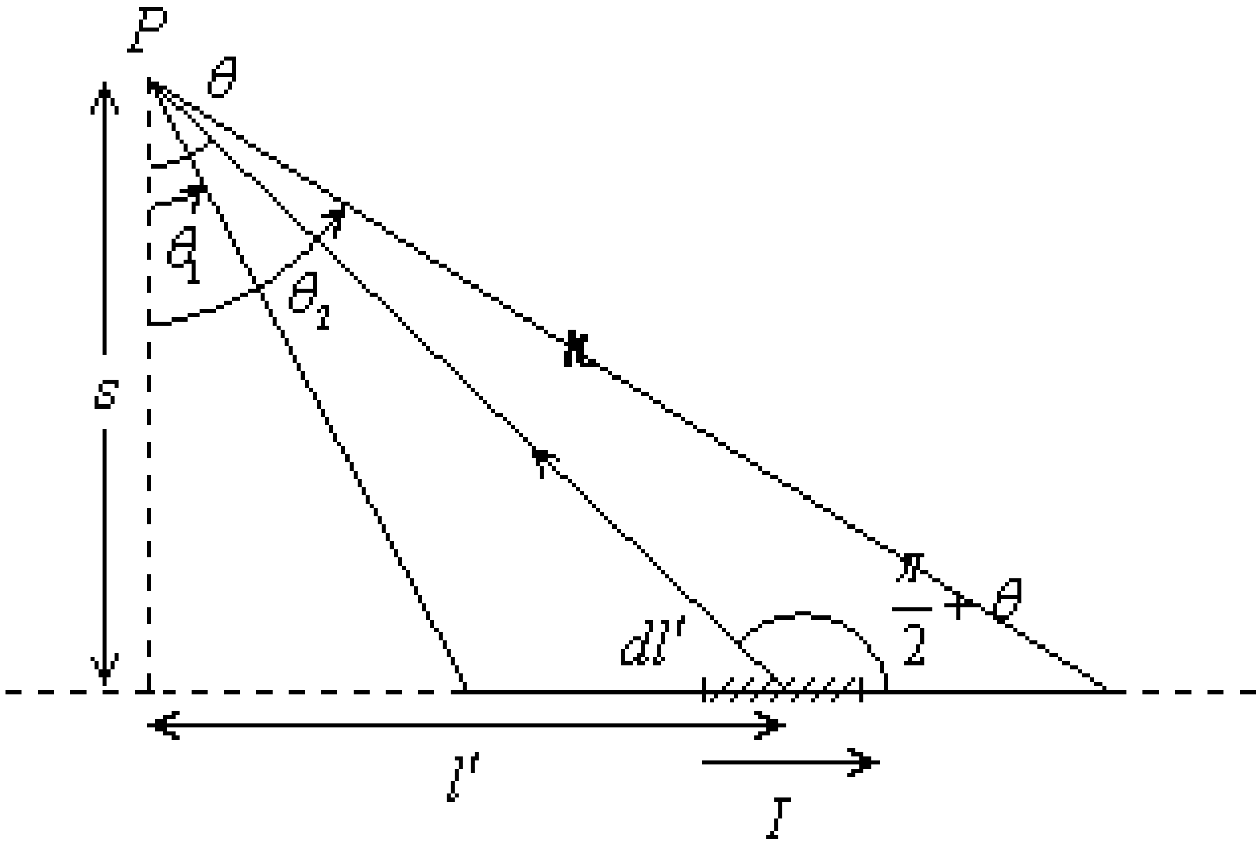
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J} \times \hat{\mathscr{r}}}{\mathscr{r}^2} d\tau'$$



$$\mathbf{I} = \mathbf{J} da_{\perp}$$

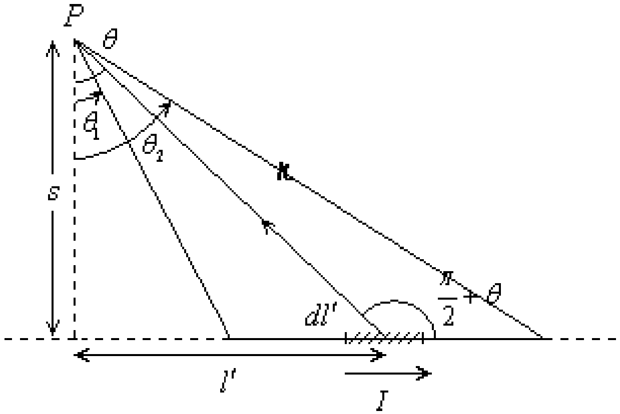
$$\mathbf{I} dl' = \mathbf{J} da_{\perp} dl' = \mathbf{J} d\tau'$$

B-field of a Straight Wire Segment



B-field of a Straight Wire Segment

Consider a wire segment as shown.
We want to calculate the field at P.



By Biot-Savart law,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

the direction of B is ⊥ to the page and points outwards.
The magnitude of the B-field is therefore,

$$B(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int \frac{dl' \sin\left(\frac{\pi}{2} + \theta\right)}{\mathscr{r}^2}$$

B-field of an Infinite Wire

Change variable from $l' \to \theta$

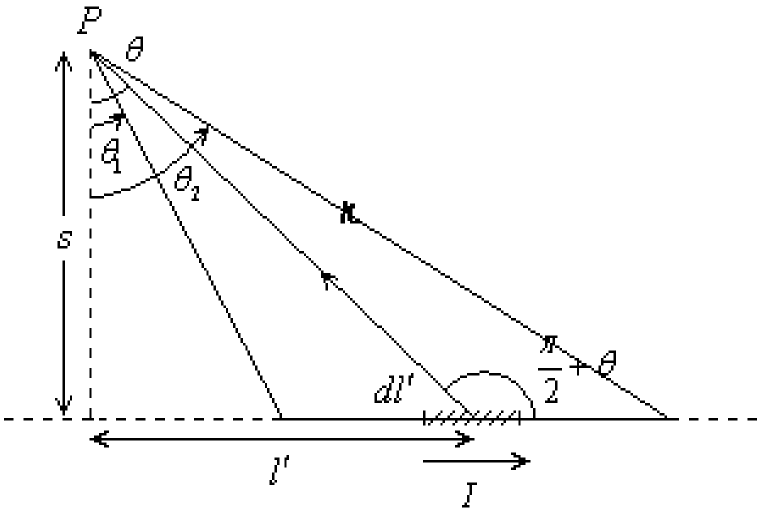
$$l' = s\tan\theta$$

$$dl' = \frac{s}{\cos^2\theta}d\theta$$

$$\frac{1}{r^2} = \frac{\cos^2\theta}{s^2}$$

$$\therefore B = \frac{\mu_0 I}{4\pi}\int_{\theta_1}^{\theta_2}\frac{\cos^2\theta}{s^2}\frac{s}{\cos^2\theta}\cos\theta d\theta$$

$$= \frac{\mu_0 I}{4\pi s}\left[\sin\theta_2 - \sin\theta_1\right]$$



For an infinitely long wire, $\theta_1 \to -\frac{\pi}{2}$, $\theta_2 \to \frac{\pi}{2}$

$$\therefore B = \frac{\mu_0 I}{2\pi s}$$

B-field of a Circular Wire

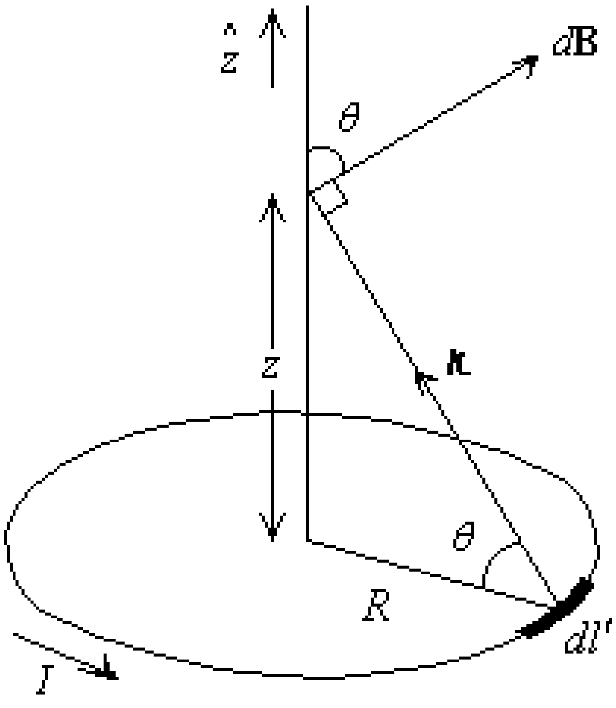
Consider a circular wire with radius R carrying a current I. Evaluate the B-field at a point directly above the center at a distance z.

By symmetry, the B-field should be along the axis, i.e. $\mathbf{B}(z) = B(z)\hat{\mathbf{z}}$

$$B(z) = \frac{\mu_0 I}{4\pi}\int \frac{d\mathbf{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2}\cdot\hat{\mathbf{z}} = \frac{\mu_0 I}{4\pi}\int \frac{dl'}{\mathscr{r}^2}\cos\theta$$

$$= \frac{\mu_0 I}{4\pi}\frac{\cos\theta}{\mathscr{r}^2}\int dl' = \frac{\mu_0 I}{4\pi}\frac{R/\mathscr{r}}{\mathscr{r}^2}2\pi R$$

$$= \frac{\mu_0 I}{2}\frac{R^2}{\left(R^2 + z^2\right)^{3/2}}$$

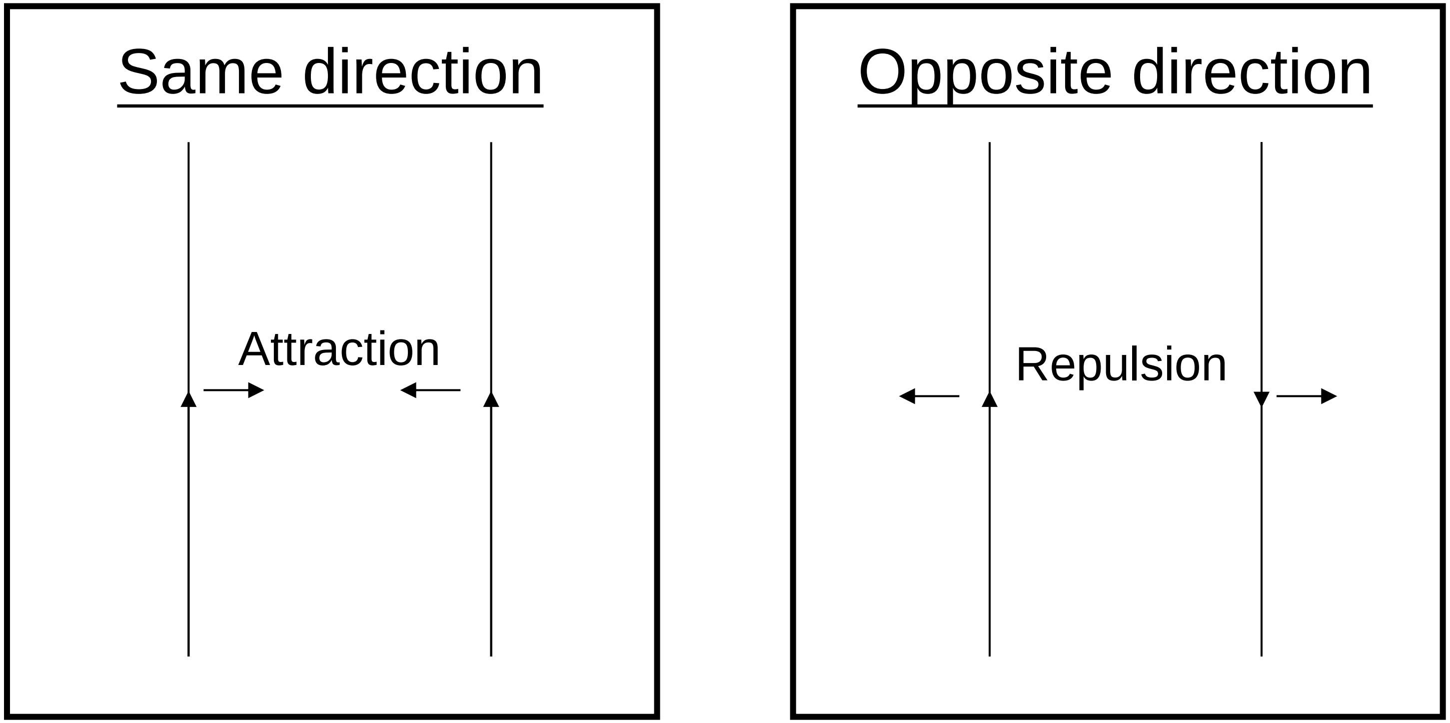


Why μ_0 is exactly $4\pi \times 10^{-7}\ \mathrm{N/A}^2$???

- Its value is so chosen by the definition of current.
- The definition of the unit of current (SI) – ampere, is related to the magnetic force between two infinitely long straight wires.

Definition of Current

- Experiments show that :



Definition of Current

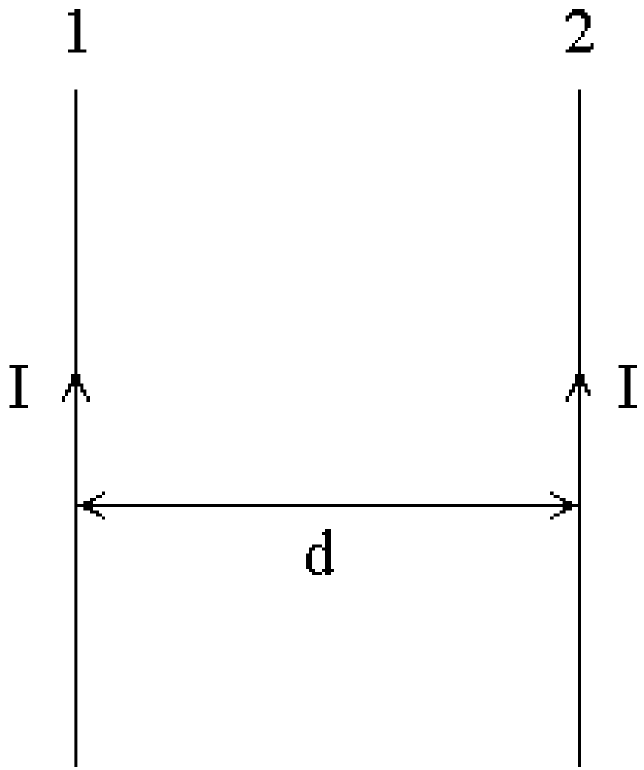
- two wires, carrying the same current, will attract each other when currents are in the same direction
- force is reversed but with the same magnitude if the currents are in opposite directions
- 1 ampere is defined as the current carried in each wire when the wires are separated by 1m and the force per unit length on each wire has a magnitude of

$$\frac{F}{l} = 2 \times 10^{-7}\,\mathrm{N/m}$$

Permeability and the Definition of Ampere

A segment of wire 2 with length l, experiences a force,

$$F = IlB$$
$$= l\frac{\mu_0 I^2}{2\pi d}$$



The force per unit length is

$$f = \frac{F}{l} = \frac{\mu_0 I^2}{2\pi d}$$

By definition, when

$$d = 1\text{m}$$
$$I = 1\text{A}$$
$$f = 2\times 10^{-7}\ \text{N/m}$$

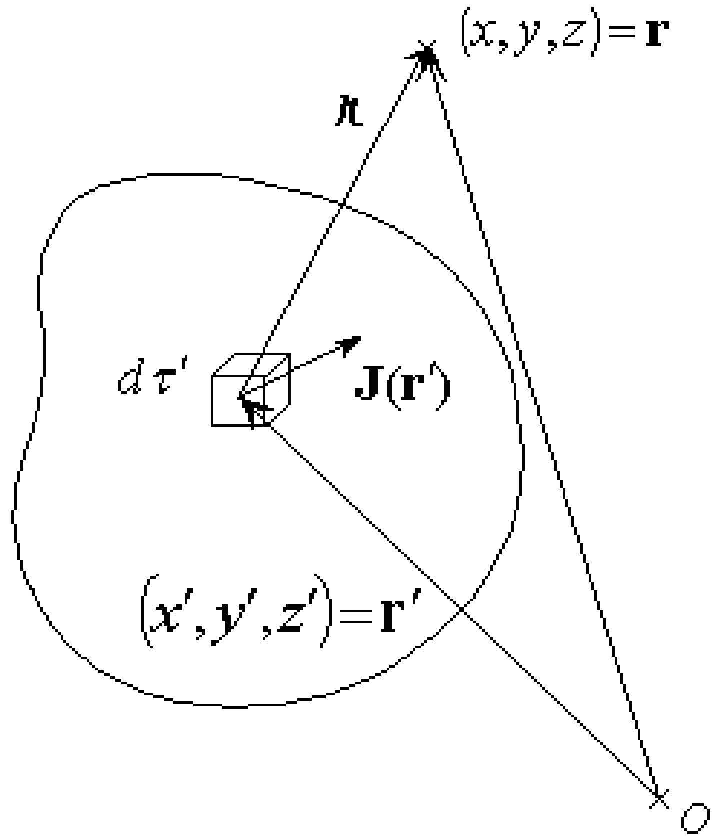
$$\therefore \mu_0 = 4\pi\times 10^{-7}\ \text{N/A}^2$$

Divergence of B Field

Consider a general current density $\mathbf{J}(\mathbf{r}') = \mathbf{J}(x', y', z')$

By Biot-Savart law,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2} \, d\tau'$$



Divergence of B

$$\nabla_{\mathbf{r}} \cdot \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla_{\mathbf{r}} \cdot \left(\mathbf{J}(\mathbf{r}') \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) d\tau'$$

$$\nabla_{\mathbf{r}} \cdot \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla_{\mathbf{r}} \cdot \left(\mathbf{J}(\mathbf{r}') \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) d\tau'$$

The del operator $\nabla_{\mathbf{r}}$ is w.r.t. $\mathbf{r}$

$$\nabla_{\mathbf{r}} \cdot \left(\mathbf{J}(\mathbf{r}') \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \cdot \left(\nabla_{\mathbf{r}} \times \mathbf{J} \right) - \mathbf{J} \cdot \left(\nabla_{\mathbf{r}} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right)$$

Since $\mathbf{J}(\mathbf{r}')$ is independent of $\mathbf{r}$

$$\nabla_{\mathbf{r}} \times \mathbf{J} = 0$$

Besides,

$$\nabla_{\mathbf{r}} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \nabla_{\mathscr{r}} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \mathbf{0}$$

(Coulomb field is curl-free.)

$$\therefore \nabla \cdot \mathbf{B} = 0$$

Integral Form

$$\because \nabla \cdot \mathbf{B} = 0$$

- By divergence theorem

$$\Rightarrow \oint_S \mathbf{B} \cdot d\mathbf{a} = 0$$

for arbitrary closed surface *S*

Curl of B Field

Ampere’s Law

$$\nabla_{\mathbf{r}} \times \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla_{\mathbf{r}} \times \left(\mathbf{J}(\mathbf{r}') \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) d\tau'$$

$$\nabla_{\mathbf{r}} \times \left(\mathbf{J}(\mathbf{r}') \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right)$$

$$= \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \cdot \nabla_{\mathbf{r}} \right) \mathbf{J}(\mathbf{r}') - \left(\mathbf{J}(\mathbf{r}') \cdot \nabla_{\mathbf{r}} \right) \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} + \mathbf{J}(\mathbf{r}') \left(\nabla_{\mathbf{r}} \cdot \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) - \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \left(\nabla_{\mathbf{r}} \cdot \mathbf{J}(\mathbf{r}') \right)$$

The 1st term and the 4th term are zero because $\mathbf{J}(\mathbf{r}')$ is independent of $\mathbf{r}$

$$\nabla_{\mathbf{r}} \times \mathbf{B} = \frac{\mu_0}{4\pi}\int \nabla_{\mathbf{r}} \times \left(\mathbf{J}(\mathbf{r}') \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) d\tau'$$

$$\nabla_{\mathbf{r}} \times \left(\mathbf{J}(\mathbf{r}') \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) = -\left(\mathbf{J}(\mathbf{r}') \cdot \nabla_{\mathbf{r}}\right)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} + \mathbf{J}(\mathbf{r}')\left(\nabla_{\mathbf{r}} \cdot \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right)$$

$$\nabla_{\mathbf{r}} \cdot \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \nabla_{\mathscr{r}} \cdot \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = 4\pi\delta^3(\mathscr{r}) = 4\pi\delta^3(\mathbf{r}-\mathbf{r}')$$

$$\therefore \nabla_{\mathbf{r}} \times \mathbf{B} = \frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{r}')4\pi\delta^3(\mathbf{r}-\mathbf{r}')d\tau' - \frac{\mu_0}{4\pi}\int \left(\mathbf{J}(\mathbf{r}') \cdot \nabla_{\mathbf{r}}\right)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} d\tau'$$

$$= \mu_0 \mathbf{J}(\mathbf{r}) - \frac{\mu_0}{4\pi}\int \left(\mathbf{J}(\mathbf{r}') \cdot \nabla_{\mathbf{r}}\right)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} d\tau'$$

$$= \mu_0 \mathbf{J}(\mathbf{r}) + \frac{\mu_0}{4\pi}\int \left(\mathbf{J}(\mathbf{r}') \cdot \nabla_{\mathbf{r}'}\right)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} d\tau'$$

$$\nabla_r \times \mathrm{B} = \mu_0 \mathbf{J}(\mathbf{r}) + \frac{\mu_0}{4\pi} \int \left(\mathbf{J}(\mathbf{r}') \cdot \nabla_{\mathbf{r}'} \right) \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} d\tau'$$

$$\left(\mathbf{J}(\mathbf{r}') \cdot \nabla_{\mathbf{r}'} \right) \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \text{ is a vector}$$

Consider the *x*-component

$$\left[\left(\mathbf{J}(\mathbf{r}') \cdot \nabla_{\mathbf{r}'} \right) \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right]_x = \left(\mathbf{J}(\mathbf{r}') \cdot \nabla_{\mathbf{r}'} \right) \frac{x - x'}{\mathscr{r}^3}$$

$$= \nabla_{\mathbf{r}'} \cdot \left(\frac{x - x'}{\mathscr{r}^3} \mathbf{J}(\mathbf{r}') \right) - \frac{x - x'}{\mathscr{r}^3} \left(\nabla_{\mathbf{r}'} \cdot \mathbf{J}(\mathbf{r}') \right)$$

From the continuity equation, $\dfrac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$

for steady state, $\dfrac{\partial \rho}{\partial t} = 0 \rightarrow \nabla_{\mathbf{r}'} \cdot \mathbf{J}(\mathbf{r}') = 0$

$$\therefore \int \left[\left(\mathbf{J}(\mathbf{r}') \cdot \nabla_{\mathbf{r}'} \right) \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right]_x d\tau' = \int \nabla_{\mathbf{r}'} \cdot \left(\frac{x - x'}{\mathscr{r}^3} \mathbf{J}(\mathbf{r}') \right) d\tau'$$

$$\therefore \int \left[\left(\mathbf{J}(\mathbf{r}') \cdot \nabla_{\mathbf{r}'} \right) \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right]_x d\tau' = \int \nabla_{\mathbf{r}'} \cdot \left(\frac{x - x'}{\mathscr{r}^3} \mathbf{J}(\mathbf{r}') \right) d\tau'$$

Use divergence theorem,

$$\int_{\mathcal{V}} \left[\left(\mathbf{J}(\mathbf{r}') \cdot \nabla_{\mathbf{r}'} \right) \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right]_x d\tau' = \oint_S \frac{x - x'}{\mathscr{r}^3} \mathbf{J}(\mathbf{r}') \cdot d\mathbf{a}'$$

When using the Biot-Savart law to evaluate the field, one must include the contributions of all current densities.

In other words, $\mathcal{V}$ includes all the currents and no current is flowing in or out at surface *S*.

$$\therefore \mathbf{J}(\mathbf{r}') \cdot d\mathbf{a}' = 0$$

The above argument obviously holds also for the other components. So

$$\int \left(\mathbf{J}(\mathbf{r}') \cdot \nabla_{\mathbf{r}'} \right) \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \, d\tau' = \mathbf{0}$$

$$\nabla_r \times \mathbf{B} = \mu_0 \mathbf{J}(\mathbf{r}) + \frac{\mu_0}{4\pi} \int \left(\mathbf{J}(\mathbf{r}') \cdot \nabla_{\mathbf{r}'} \right) \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \, d\tau'$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad \longleftarrow \quad \textit{Ampere's Law}$$

Integral Form of Ampere's Law

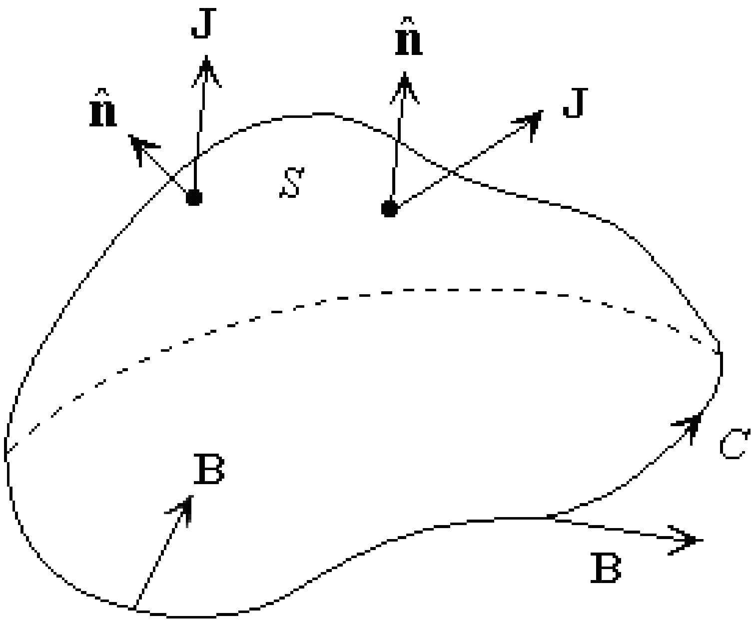
Consider a surface *S* with *C* as the boundary.

Stokes' thm:

$$\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \oint_C \mathbf{B} \cdot d\mathbf{l}$$

$$\therefore \quad \oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{a}$$

$I_{\text{enc}} = \int_S \mathbf{J} \cdot d\mathbf{a}$ is the amount of current enclosed by *C*



$$\therefore \quad \oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$$

Ampere's law in integral form

Magnetostatics

Application of Ampere’s Law

Application of Ampere's Law

Like the Gauss's law, the Ampere's law can be used to evaluate the B-field easily when the system exhibits certain symmetries.

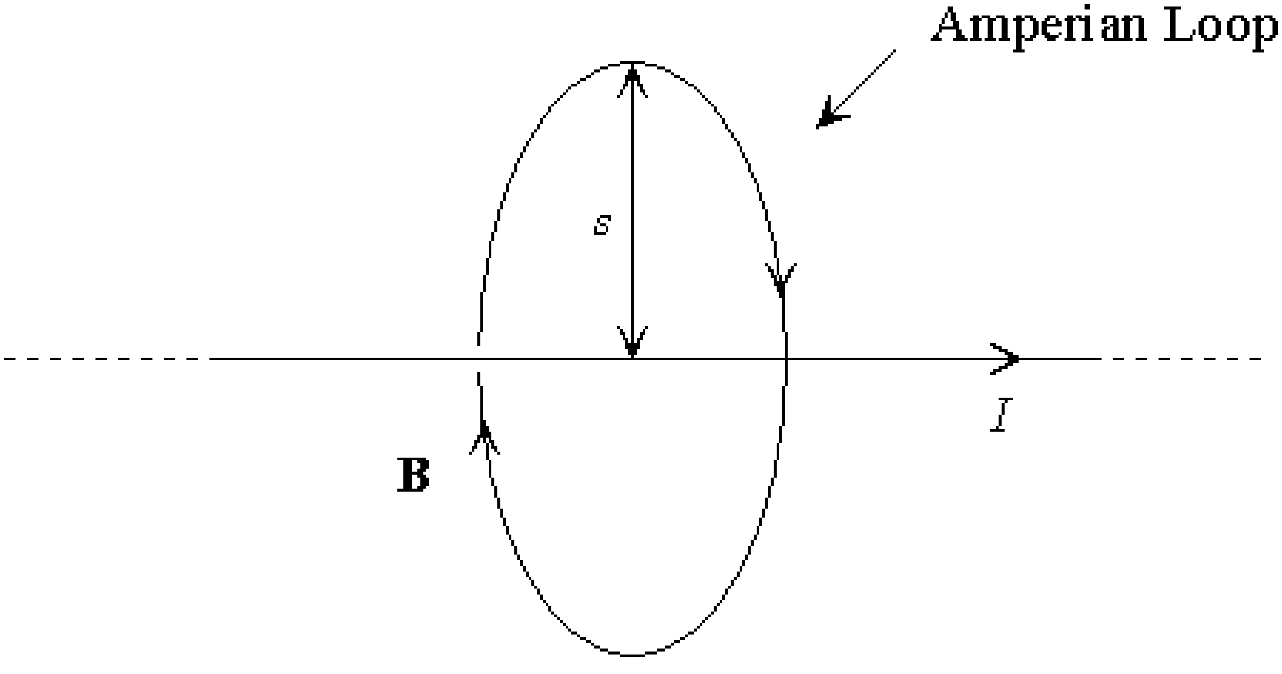
In this case, one will usually find the ampere's law in integral form more useful.

Ampere's law in integral form:

$$\int_{c} \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\mathrm{enc}}$$

Example:

Use Ampere's law to find the B-field of an infinity long wire carrying a current ***I***.



Solution:

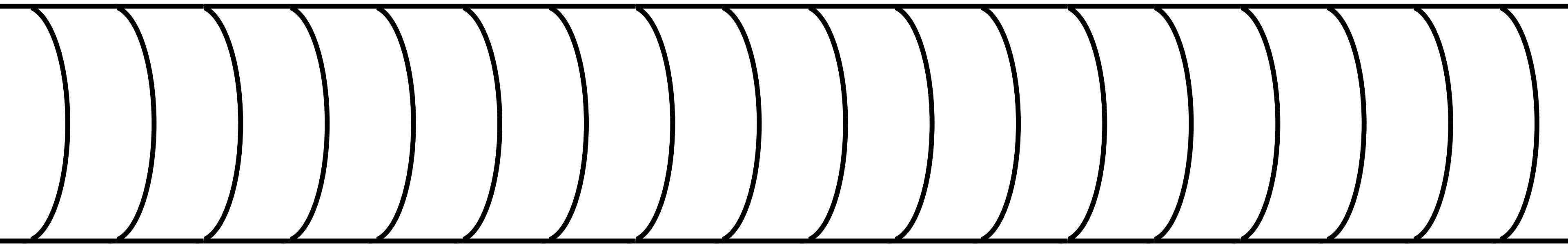
From Biot-Savart law and right-hand rule, the direction of B-field is circumferential. By symmetry, its magnitude is a constant on the amperian loop. Apply Ampere's law:

$$\oint \mathbf{B} \cdot d\mathbf{l} = B \cdot 2\pi s = \mu_0 I$$

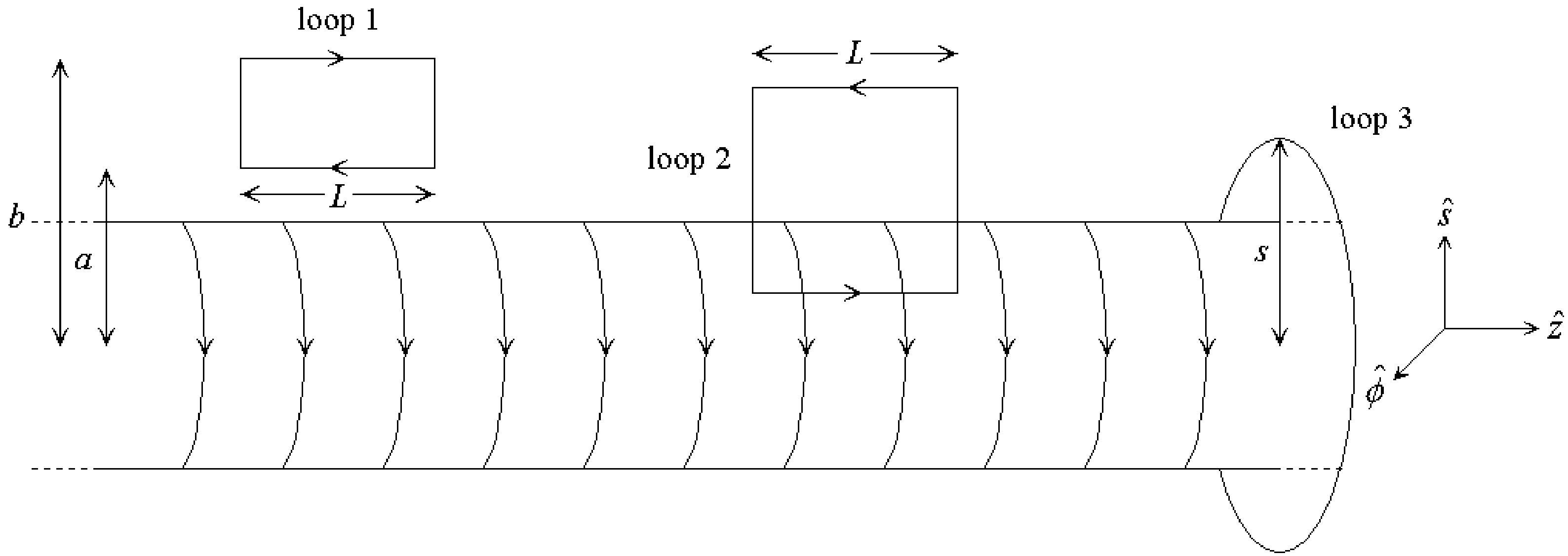
$$\therefore \quad \mathrm{B} = \frac{\mu_0 I}{2\pi s}$$

Example:

Use Ampere's law to find the B-field of an infinitely long solenoid carrying a current I.



Solution: Use cylindrical coordinate



By rotational and translational symmetry, the field depends only on *s*. Consider the circular amperian loop (loop3).

By ampere's law: $$B_\phi \cdot 2\pi s = \mu_0 I_{\text{enc}} = 0$$

$$\therefore \qquad \mathrm{B}_\phi = 0$$

The radial component B_s is also zero.

If you flip the solenoid to the opposite direction,
B_s is unchanged.

But flipping the solenoid is equivalent to switching the current to flow in opposite direction, and hence $B_s \to -B_s$

$$\therefore \quad B_s = 0$$

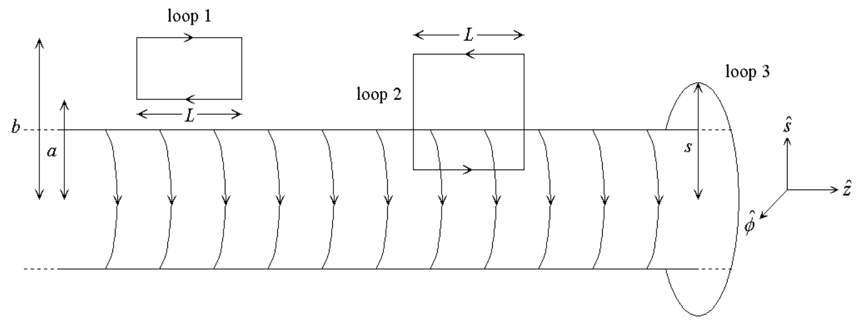
(One can also argue by using $\nabla \cdot \mathbf{B} = 0$).

In conclusion, $\mathbf{B}$ only has $\hat{\mathbf{z}}$ component, and its magnitude depends on *s* only: $$\mathbf{B} = B(s)\hat{\mathbf{z}}$$

Consider a rectangular amperian loop (loop 1) outside the solenoid. Apply Ampere's law:

$$B_z(b)L - B_z(a)L = \mu_0 I_{\text{enc}} = 0$$

$$\therefore \quad B_z(b) = B_z(a)$$

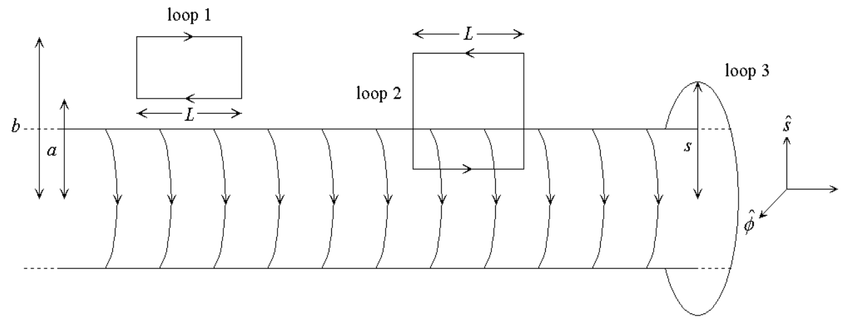


This is true for all *a, b* > radius of the solenoid

$\therefore \mathbf{B}$ is constant outside the solenoid

But $\mathbf{B} \to \mathbf{0}$ as $s \to \infty$

$\therefore \quad \mathbf{B} = \mathbf{0}$ outside the solenoid.



To find the field inside, consider the amperian loop 2. From Ampere's law,

$$BL = \mu_0 I_{\text{enc}}$$

where *B* is the magnitude of the field at the bottom edge of the loop. If the number of the turns of wire per unit length is *n*, then

$$BL = \mu_0 nIL$$

$$\therefore\ B = \mu_0 nI$$

which is a constant.

$\therefore\ \mathbf{B} = \mu_0 nI\ \hat{\mathbf{z}}$ is a constant inside the solenoid, pointing to the direction determined by right hand rule.

Magnetostatics

Magnetic Vector Potential

Magnetic Vector Potential

Since $\nabla \cdot \mathbf{B} = 0$, we can define a vector potential of $\mathbf{B}$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$\mathbf{A}$ is called the vector potential because the divergence of a curl is always zero, hence

$\nabla \cdot \mathbf{B} = 0$ is satisfied automatically.

Magnetic Vector Potential

The one left is then the Ampere's law:

$\Rightarrow$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J}$$

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$$

$\mathbf{A}$ is not uniquely defined by its definition. You can add the gradient of a scalar to the vector potential without changing its curl:

$$\mathbf{A}' = \mathbf{A} + \nabla \lambda$$

$$\nabla \times \mathbf{A}' = \nabla \times \mathbf{A} + \nabla \times (\nabla \lambda)$$

$$= \nabla \times \mathbf{A}$$

Magnetic Vector Potential

Suppose we have found a particular vector potential $\mathbf{A}_0$, we want to find λ so that:

$$\mathbf{A} = \mathbf{A}_0 + \nabla\lambda \quad \text{is divergence-free}$$

i.e.,

$$\nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{A}_0 + \nabla \cdot \left(\nabla\lambda\right) = 0$$

$$\Rightarrow \quad \nabla^2\lambda = -\nabla \cdot \mathbf{A}_0$$

This is just the mathematical expression of the Poisson equation with $\nabla \cdot \mathbf{A}_0$ replacing ρ as the source.

The Poisson equation provides always a solution for λ!

For example:

If $\nabla \cdot \mathbf{A}_0 \rightarrow 0$ at infinity, we know that the solution is

$$\lambda = \frac{1}{4\pi}\int \frac{\nabla \cdot \mathbf{A}_0}{r}\, d\tau'$$

Magnetic Vector Potential

Now, since $\mathbf{A}$ is divergence free, the Ampere's law implies

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

A set of 3 Poisson equations, one for each vector component!

Assuming that in a particular system:

$\mathbf{J} \rightarrow \mathbf{0}$ at infinity,

then the solution becomes

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}} d\tau'$$

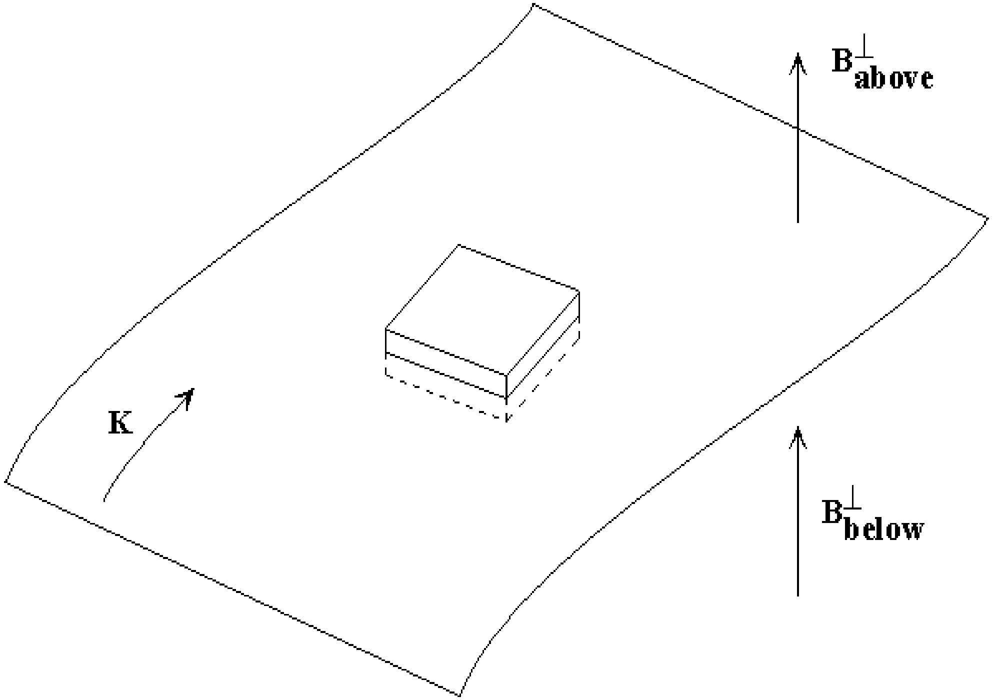
Magnetostatics

Magnetostatic Boundary Conditions

Magnetostatic Boundary Conditions

The magnetic field is discontinuous across a surface current. The relation between the fields on both sides can derived by using

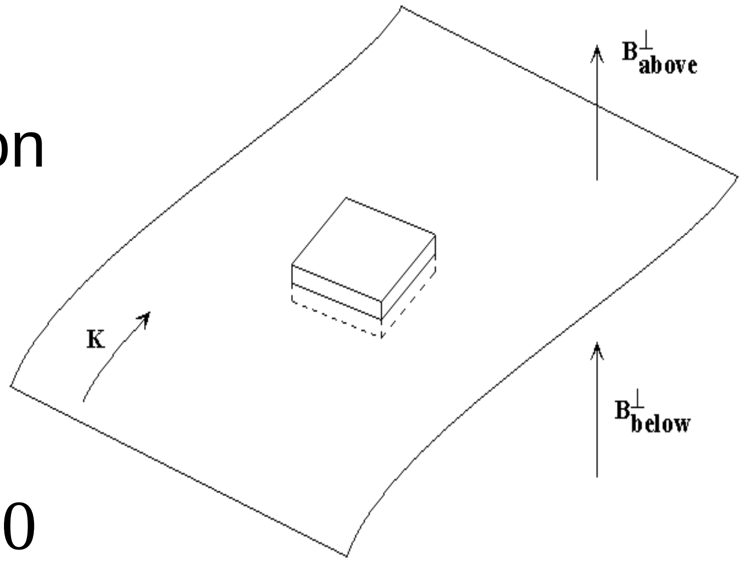
$$\begin{cases} \oint_s \mathbf{B} \cdot d\mathbf{a} = 0 \\ \oint_c \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{enc} \end{cases}$$



Magnetostatic Boundary Conditions

For the perpendicular component, consider the small pill-box and use

$$\oint_S \mathbf{B} \cdot d\mathbf{a} = 0$$



The pill-box is so thin that the flux on the side-edges can be neglected.

Let the area of the top and bottom faces of the pill-box be *A*.

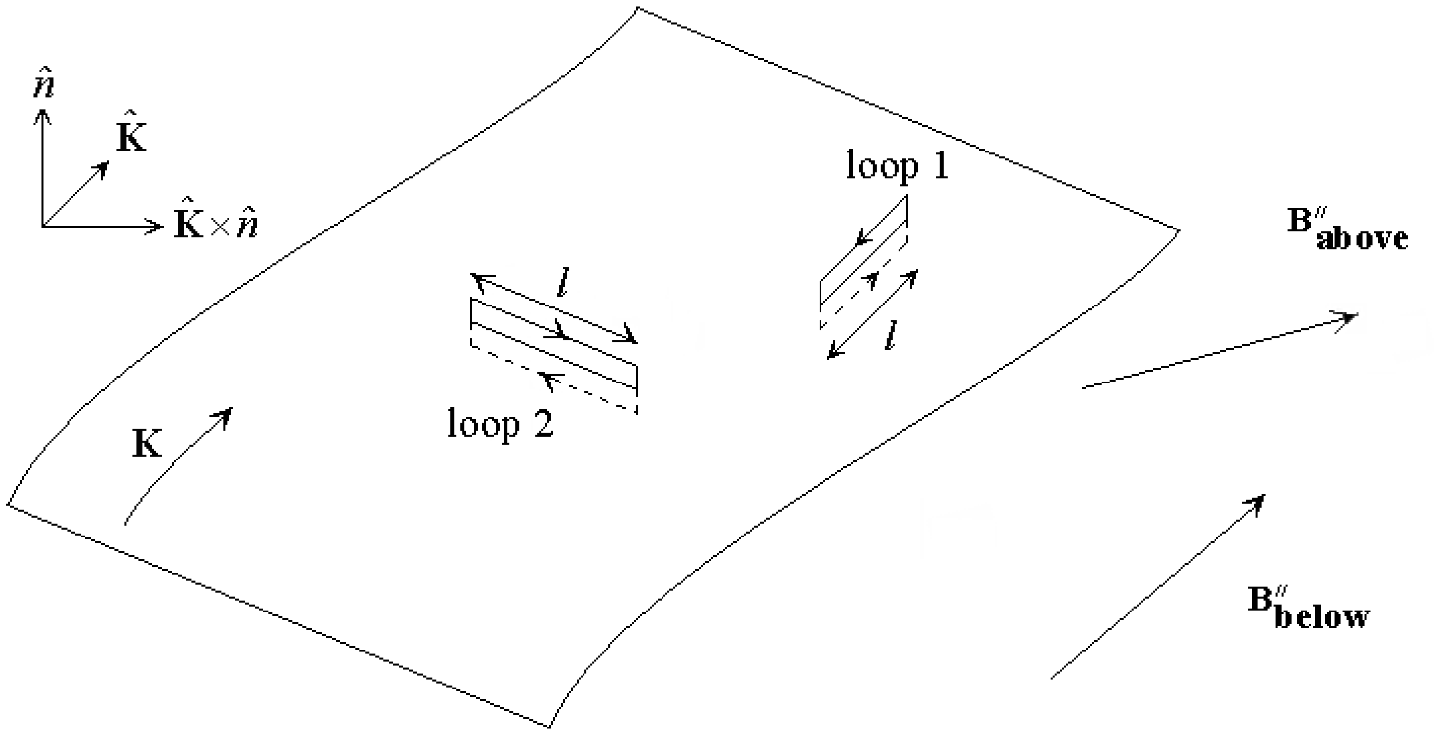
Then $$B^{\perp}_{\text{above}} A - B^{\perp}_{\text{below}} A = 0$$

$$\Rightarrow \quad B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}}$$

∴The perpendicular component of the B-field is continuous.

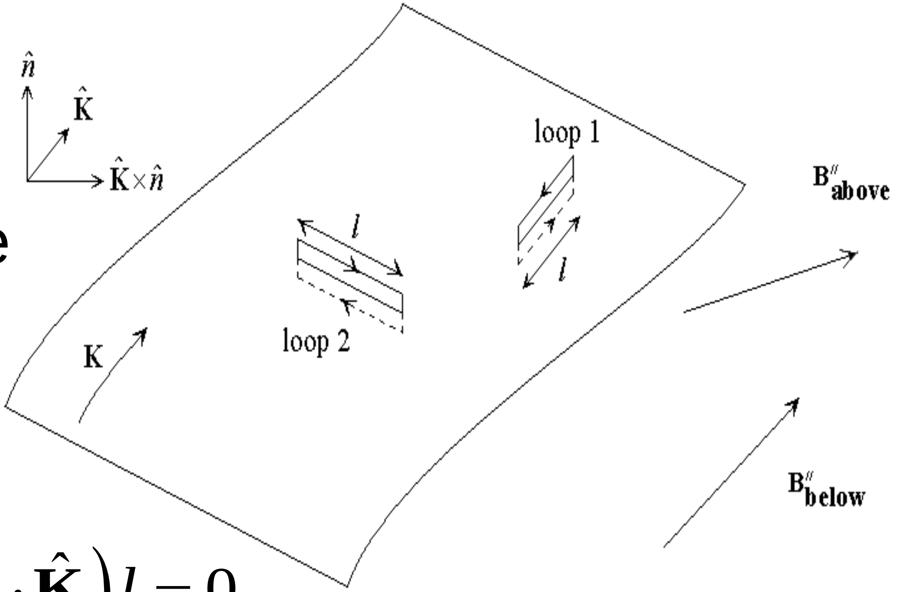
Magnetostatic Boundary Conditions

For the parallel component, consider a very "thin" rectangular amperian loop across the surface.



Magnetostatic Boundary Conditions

Let the unit vector along the direction of the current be $\hat{\mathbf{K}}$ and $\hat{\mathbf{K}} \times \hat{\mathbf{n}}$



Consider the amperian loop 1, of which the sides are along the $\hat{\mathbf{K}}$ direction. This loop does not enclose any current.

So, by Ampere's law

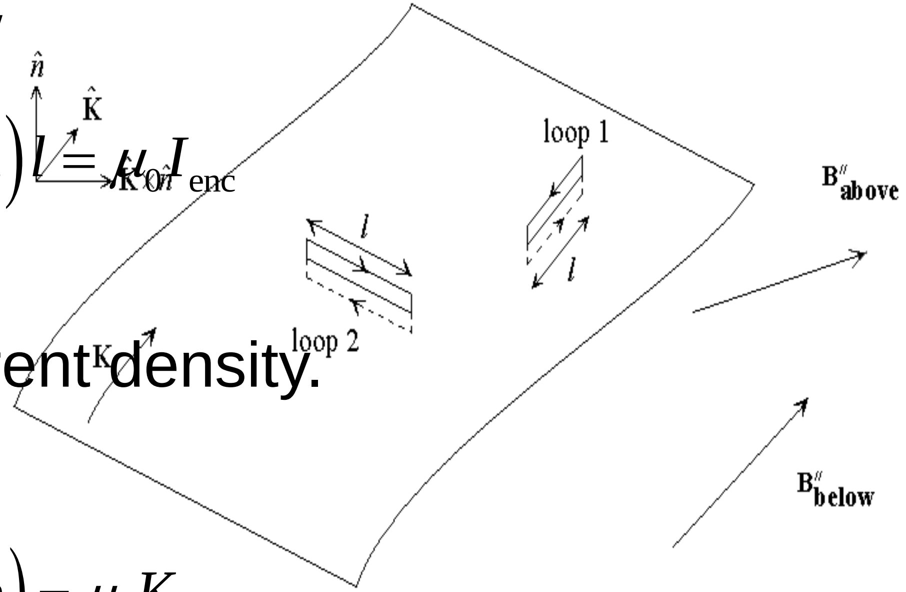
$$\left(-\mathbf{B}^{//}_{\text{above}} \cdot \hat{\mathbf{K}}\right) l + \left(\mathbf{B}^{//}_{\text{below}} \cdot \hat{\mathbf{K}}\right) l = 0$$

$$\therefore \qquad \mathbf{B}^{//}_{\text{above}} \cdot \hat{\mathbf{K}} = \mathbf{B}^{//}_{\text{below}} \cdot \hat{\mathbf{K}}$$

Magnetostatic Boundary Conditions

Now, consider the Amperian loop 2, with sides perpendicular to the current. By Ampere's law

$$\mathbf{B}^{//}_{\text{above}} \cdot \left(\hat{\mathbf{K}} \times \hat{\mathbf{n}}\right) l - \mathbf{B}^{//}_{\text{below}} \cdot \left(\hat{\mathbf{K}} \times \hat{\mathbf{n}}\right) l = \mu_0 I_{\text{enc}}$$



By the definition of surface current density.

$$I_{\text{enc}} = Kl$$

$$\therefore \quad \mathbf{B}^{//}_{\text{above}} \cdot \left(\hat{\mathbf{K}} \times \hat{\mathbf{n}}\right) - \mathbf{B}^{//}_{\text{below}} \cdot \left(\hat{\mathbf{K}} \times \hat{\mathbf{n}}\right) = \mu_0 K$$

Magnetostatic Boundary Conditions

Since

$$\mathbf{B}_{\text{above}} \cdot \hat{\mathbf{n}} = \mathbf{B}_{\text{below}} \cdot \hat{\mathbf{n}}$$

$$\mathbf{B}_{\text{above}} \cdot \hat{\mathbf{K}} = \mathbf{B}_{\text{below}} \cdot \hat{\mathbf{K}}$$

$$\mathbf{B}_{\text{above}} \cdot (\hat{\mathbf{K}} \times \hat{\mathbf{n}}) = \mathbf{B}_{\text{below}} \cdot (\hat{\mathbf{K}} \times \hat{\mathbf{n}}) + \mu_0 K$$

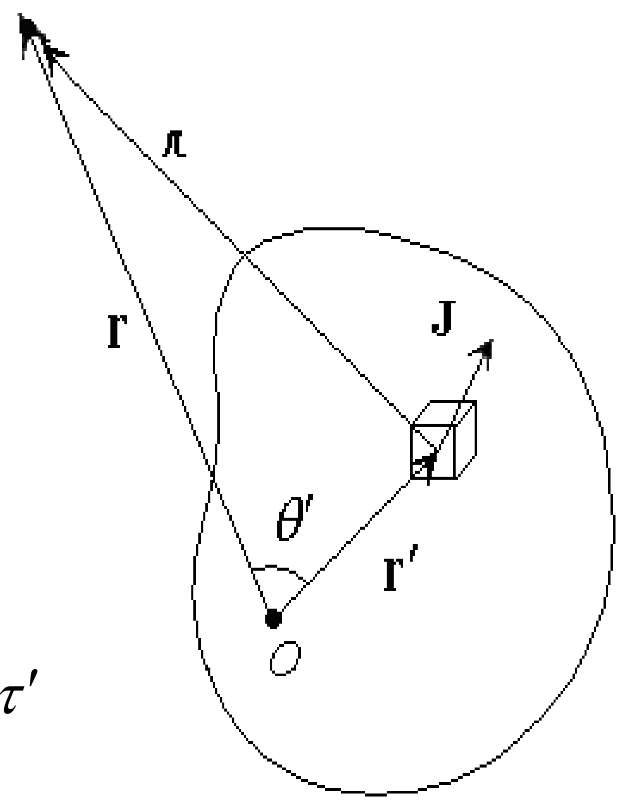
$$\therefore \quad \mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0 K \hat{\mathbf{K}} \times \hat{\mathbf{n}} = \mu_0 \mathbf{K} \times \hat{\mathbf{n}}$$

Magnetostatics

Multipole Expansion of the Vector Potential

Similar to the multipole expansion of the scalar potential *V* in electrostatics, we make use of the relation

$$\frac{1}{\mathsf{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^{n} P_n(\cos\theta')$$



Consider a localized current distribution as shown.

$$\therefore \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{\mathsf{r}}\, d\tau'$$

$$= \frac{\mu_0}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int \mathbf{J}(\mathbf{r}')(r')^{n} P_n(\cos\theta')\, d\tau'$$

- To avoid tedious mathematics, we shall consider, instead of a general volume current density **J**, a linear current flowing in a localized wire with uniform current *I*:

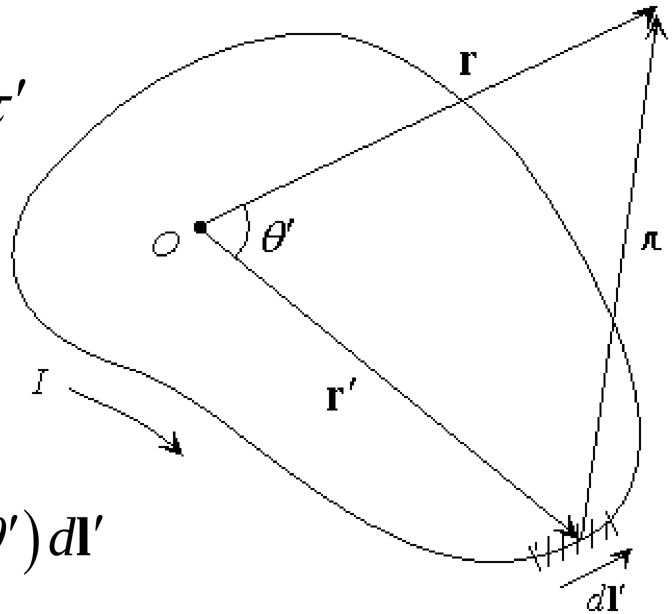
$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}}\,d\tau'$$

$$= \frac{\mu_0}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int \mathbf{J}(\mathbf{r}')(r')^n P_n(\cos\theta')\,d\tau'$$

$$\downarrow$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}}{\mathscr{r}}\,dl'$$

$$= \frac{\mu_0 I}{4\pi}\int \frac{1}{\mathscr{r}}\,d\mathbf{l}' = \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int r'^n P_n(\cos\theta')\,d\mathbf{l}'$$

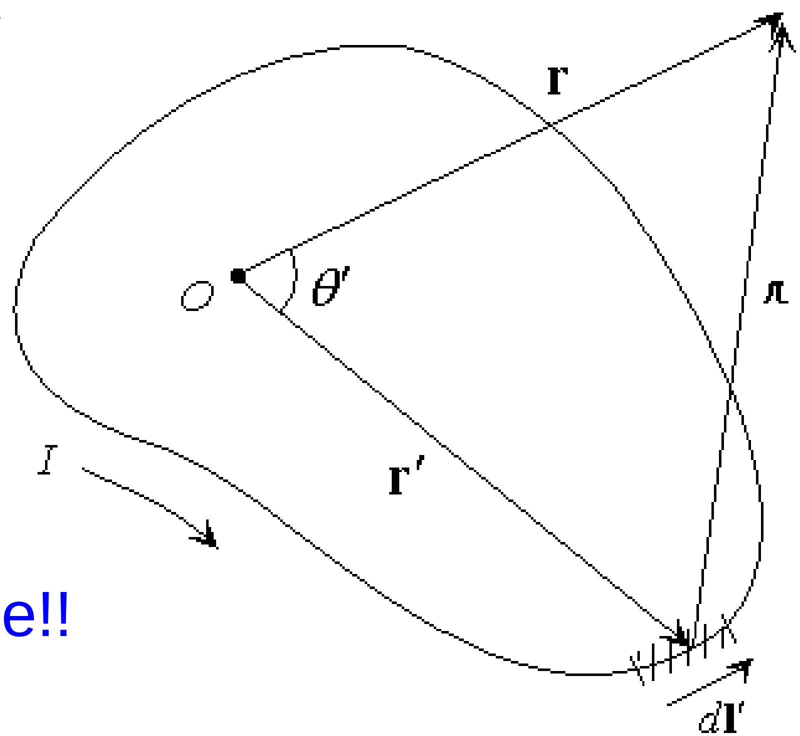


$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}}{\mathrm{r}} dl' = \frac{\mu_0 I}{4\pi}\int \frac{1}{\mathrm{r}} d\mathbf{l}'$$
$$= \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\theta')d\mathbf{l}'$$

Monopole: $n = 0$

$$\mathbf{A}_{\text{mon}}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\frac{1}{r}\oint d\mathbf{l}'$$

$$\oint d\mathbf{l}' = \mathbf{0}$$



For a closed loop

$$\therefore \quad \mathbf{A}_{\text{mon}}(\mathbf{r}) = \mathbf{0}$$

There is no magnetic monopole!!

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}}{\text{r}} dl' = \frac{\mu_0 I}{4\pi}\int \frac{1}{\text{r}} d\mathbf{l}'$$
$$= \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\theta') d\mathbf{l}'$$

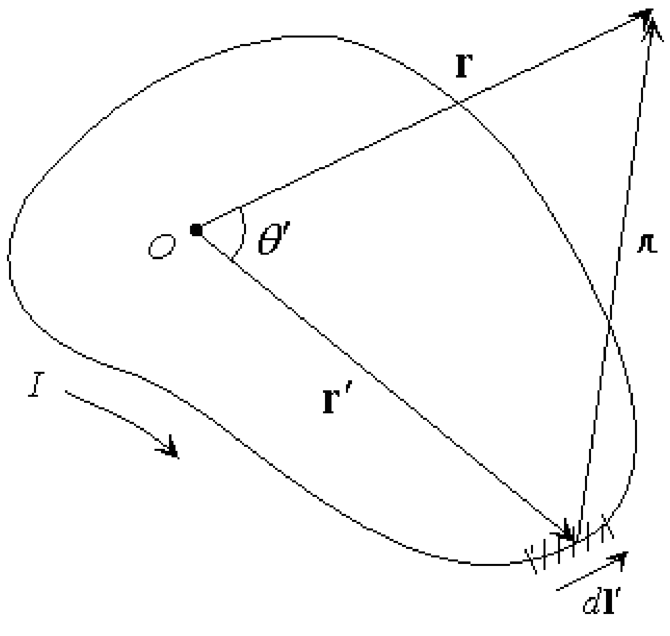
Dipole: $n = 1$

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\frac{1}{r^2}\oint r'\cos\theta \, d\mathbf{l}'$$
$$= \frac{\mu_0 I}{4\pi r^2}\oint (\hat{\mathbf{r}}\cdot\mathbf{r}') \, d\mathbf{l}'$$

It can be shown that (recall assignment 1)

$$\oint (\hat{\mathbf{r}}\cdot\mathbf{r}') \, d\mathbf{l}' = -\hat{\mathbf{r}}\times\int d\mathbf{a}'$$

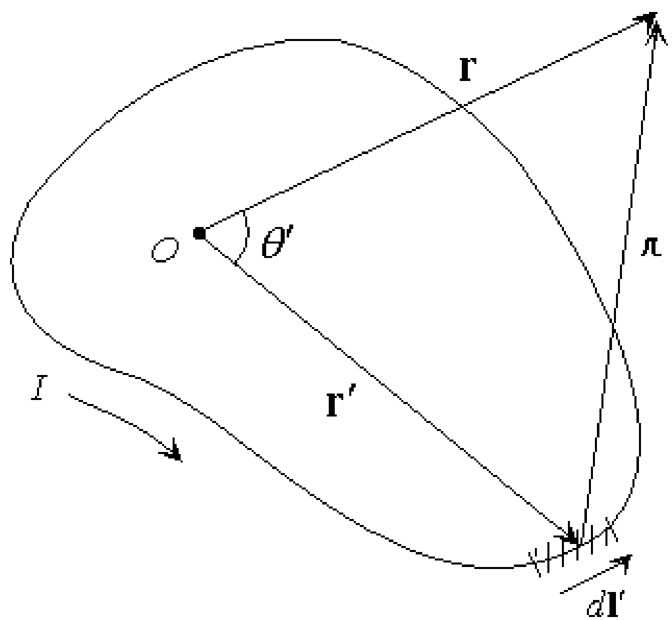
$$\therefore \quad \mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi r^2}\left(I\int d\mathbf{a}'\right)\times\hat{\mathbf{r}}$$



$$\oint (\hat{\mathbf{r}} \cdot \mathbf{r}') \, d\mathbf{l}' = -\hat{\mathbf{r}} \times \int d\mathbf{a}'$$

$$\therefore \quad \mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi r^2} \left(I \int d\mathbf{a}' \right) \times \hat{\mathbf{r}}$$

Define $\mathbf{a} = \int d\mathbf{a}'$ as the vector area of the loop and $\mathbf{m} = I\mathbf{a}$ as the magnetic dipole moment.



$$\oint (\hat{\mathbf{r}} \cdot \mathbf{r}') \, d\mathbf{l}' = -\hat{\mathbf{r}} \times \int d\mathbf{a}'$$

$$\therefore \quad \mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi r^2} \left(I \int d\mathbf{a}' \right) \times \hat{\mathbf{r}}$$

$\mathbf{m} = I\mathbf{a}$ is the magnetic dipole moment.

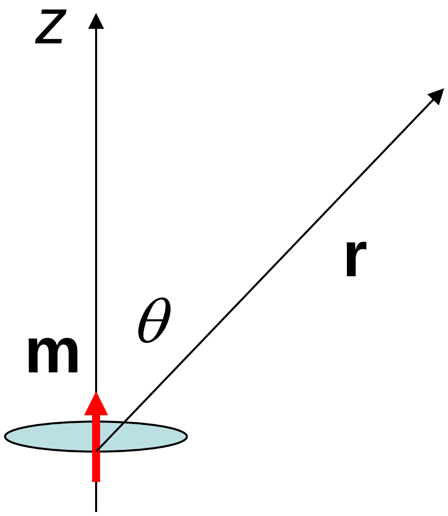
$$\therefore \; \mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}$$

The dipole field can be evaluated by

$$\mathbf{B}_{\text{dip}} = \nabla \times \mathbf{A}_{\text{dip}}$$

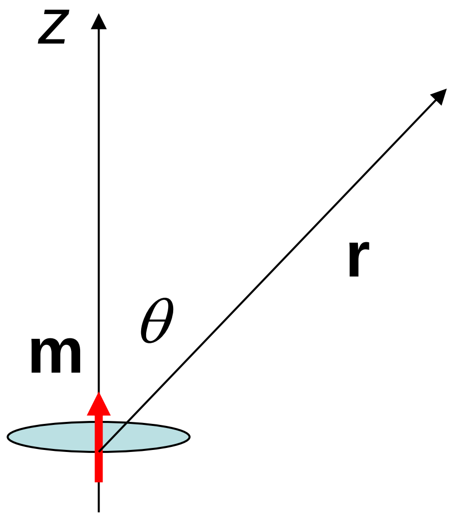
$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}$$

In spherical coordinate with $\mathbf{m}$ at the origin pointing along $\hat{\mathbf{z}}$ we have



$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\boldsymbol{\phi}}$$

$$\mathbf{A}_{\rm dip}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\boldsymbol{\phi}}$$



$$\mathbf{B}_{\rm dip} = \nabla\times\mathbf{A}_{\rm dip}$$

$$= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi\right) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}\left(rA_\phi\right)\right]\hat{\boldsymbol{\theta}}$$

$$+\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

$$= \hat{\mathbf{r}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi\right) - \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial r}\left(rA_\phi\right)$$

$$= \hat{\mathbf{r}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\right) - \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\right)$$

$$= \frac{\mu_0 m}{4\pi}\left[\hat{\mathbf{r}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin^2\theta}{r^2}\right) - \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\sin\theta}{r}\right)\right]$$

$$= \frac{\mu_0 m}{4\pi}\left[\hat{\mathbf{r}}\frac{1}{r\sin\theta}\frac{2\sin\theta\cos\theta}{r^2} - \hat{\boldsymbol{\theta}}\frac{1}{r}\left(-\frac{\sin\theta}{r^2}\right)\right]$$

$$\therefore\quad \mathbf{B}_{\rm dip}(\mathbf{r}) = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right)$$

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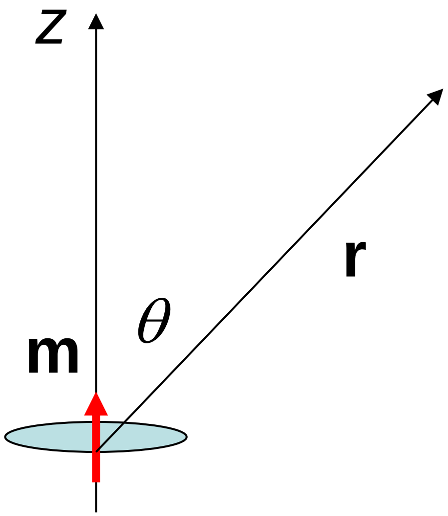

$$\mathbf{B}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0 m}{4\pi r^3}\left(3\cos\theta\,\hat{\mathbf{r}} - \cos\theta\,\hat{\mathbf{r}} + \sin\theta\;\hat{\boldsymbol{\theta}}\right)$$

$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\;\hat{\boldsymbol{\theta}}$$

$$\mathbf{B}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi r^3}\left(3m\cos\theta\,\hat{\mathbf{r}} - m\hat{\mathbf{z}}\right)$$

In coordinate-free form: $$\mathbf{B}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\right]$$