Special Techniques for Calculating Potentials

Laplace’s Equation

Recall:

Laplace's equation:

$$\nabla^2 V = \theta \frac{1}{\varepsilon_0} \rho$$

The solutions of the Laplace's equation are called the **harmonic functions**

Laplace's equation:

- Cartesian 1D:

$$\frac{d^2V}{dx^2} = 0$$

- Cartesian 2D:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

- Cartesian 3D:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Laplace's Equation

- Spherical coordinates:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0$$

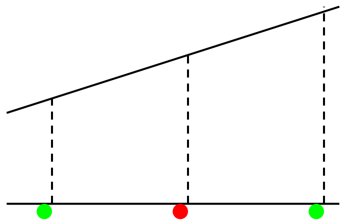
- Cylindrical coordinates:

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right)+\frac{1}{s^2}\frac{\partial^2 V}{\partial\phi^2}+\frac{\partial^2 V}{\partial z^2}=0$$

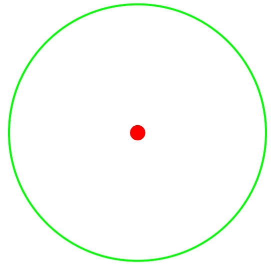
Properties of Harmonic Functions I

1. $V(\mathbf{r}_0)$ is the average of *V* of all points at a certain distance from $\mathbf{r}_0$

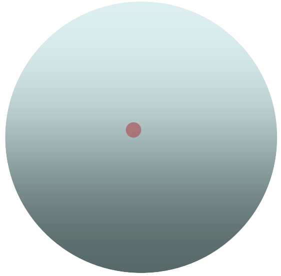
- 1D: $$V(x_0) = \frac{1}{2}\left[V(x_0 + R) + V(x_0 - R)\right]$$



- 2D: $$V(x_0, y_0) = \frac{1}{2\pi R}\oint_{\text{circle}} V\,dl$$



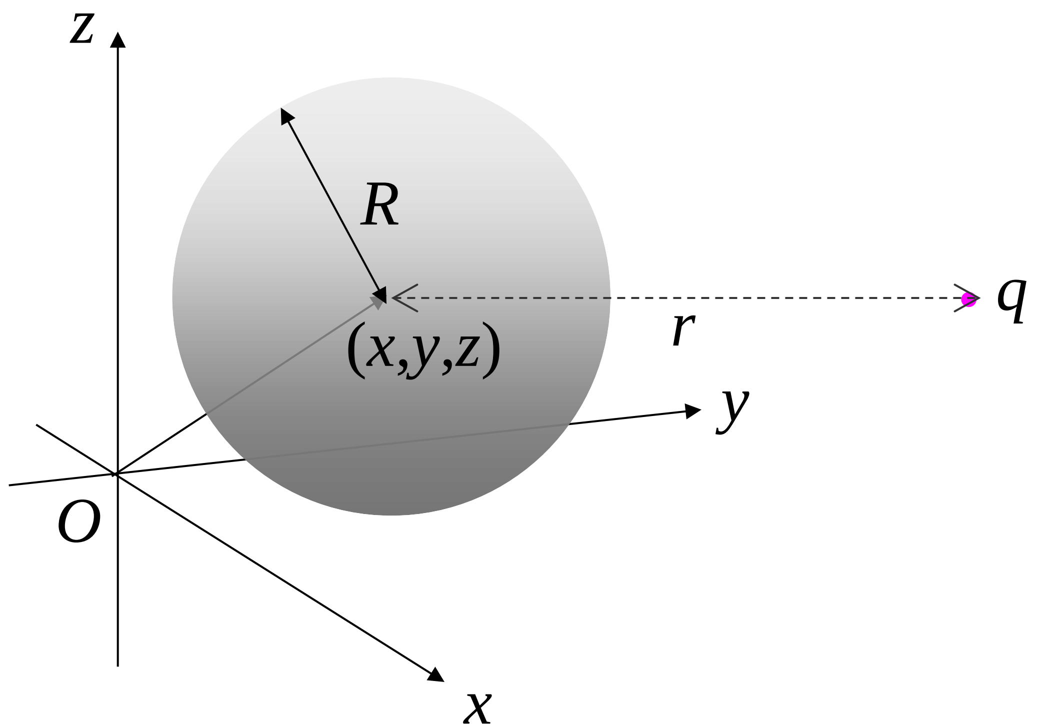
- 3D: $$V(x_0, y_0, z_0) = \frac{1}{4\pi R^2}\oint_{\text{sphere}} V\,da$$

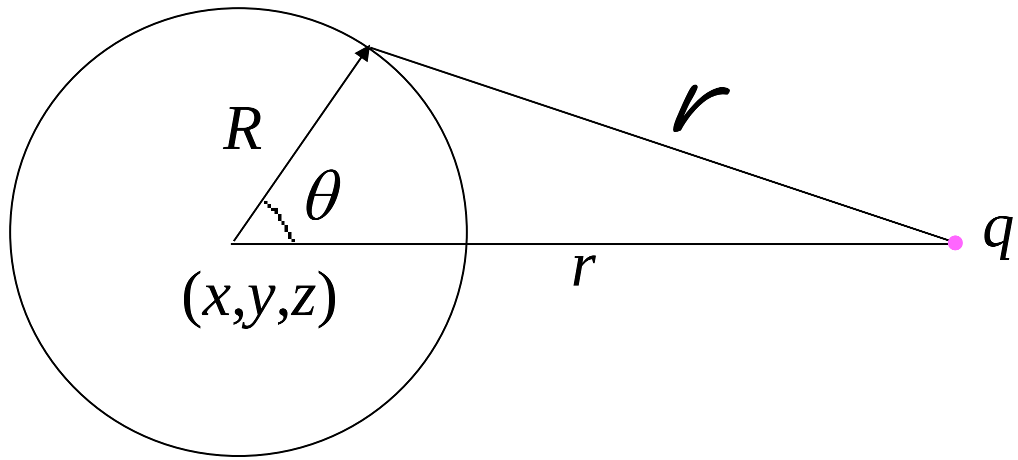


Proof of (1) for 3D case:

Since we are considering charge-free regions, the potential must be due to some charge distributions outside the region.

Consider a point charge q at a distance r from (x,y,z), where $r > R$





The potential at a point on the spherical surface shown in the figure is

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{r^2 + R^2 - 2rR\cos\theta}}$$

The average over the spherical surface is therefore

$$\langle V \rangle_S = \frac{1}{4\pi R^2} \frac{q}{4\pi\varepsilon_0} \int \left(r^2 + R^2 - 2rR\cos\theta \right)^{-1/2} R^2 \sin\theta \, d\theta \, d\phi$$

$$= \frac{q}{16\pi^2 \varepsilon_0} \int_0^{2\pi} d\phi \int_0^{\pi} \left(r^2 + R^2 - 2rR\cos\theta \right)^{-1/2} \sin\theta \, d\theta$$

$$= \frac{q}{16\pi^2 \varepsilon_0} \times 2\pi \times \int_{-1}^{1} \left(r^2 + R^2 - 2rRw \right)^{-1/2} dw \qquad \left(w = \cos\theta \right)$$

$$= \frac{q}{8\pi\varepsilon_0} \left[-\frac{\sqrt{r^2 + R^2 - 2rRw}}{rR} \right]_{-1}^{1}$$

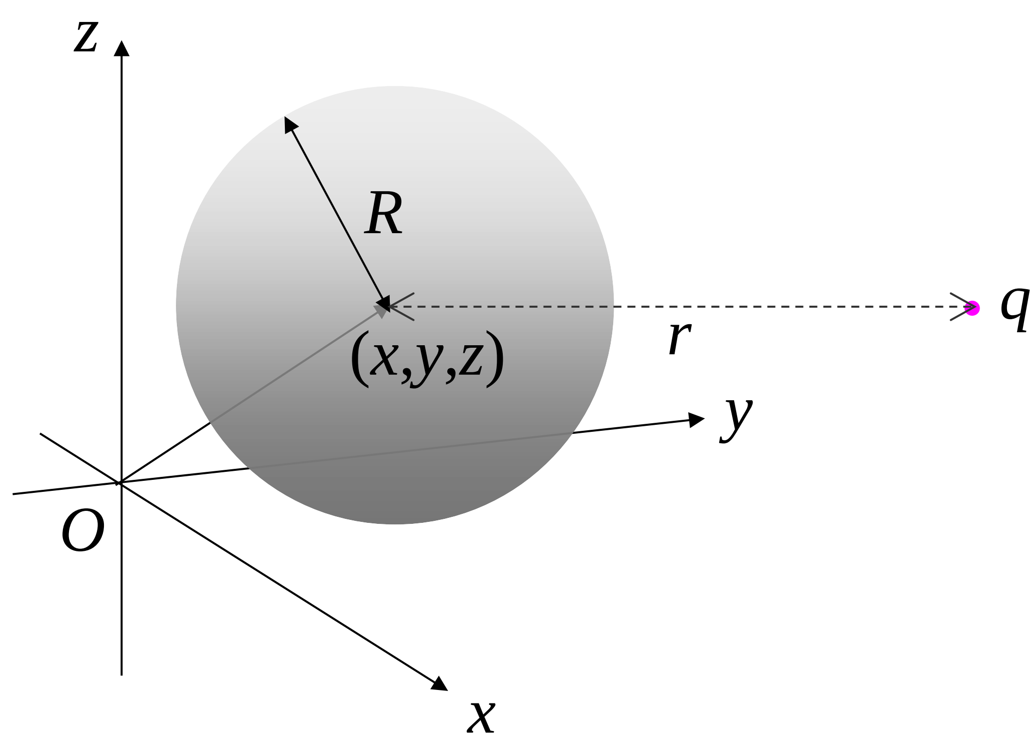
$$\langle V \rangle_S = \frac{q}{8\pi\varepsilon_0 rR}\left[\sqrt{r^2 + R^2 + 2rR} - \sqrt{r^2 + R^2 - 2rR}\right]$$

$$= \frac{q}{8\pi\varepsilon_0 rR}\left[\left(r + R\right) - \left(r - R\right)\right] \qquad (\because r > R)$$

$$= \frac{q}{4\pi\varepsilon_0 r}$$

$$= V\left(x, y, z\right)$$

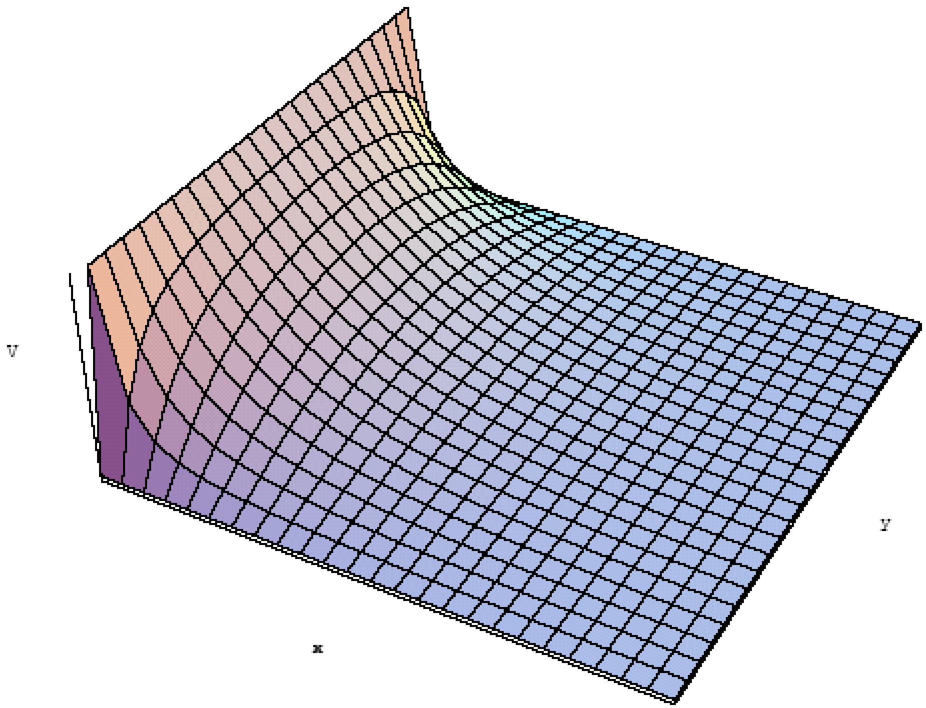
By the principle of superposition, the potential due to any charge distribution outside the sphere satisfies (1).



Properties of Harmonic Functions II

2. By (1), it can be shown that for harmonic functions, there is no local maxima or minima

   Extreme values of *V* must occur at the boundaries



Proof

- If V has a local maximum or minimum at an interior point $\mathbf{P}(x,y,z)$,
- then there exists a neighborhood of $\mathbf{P}$ in which

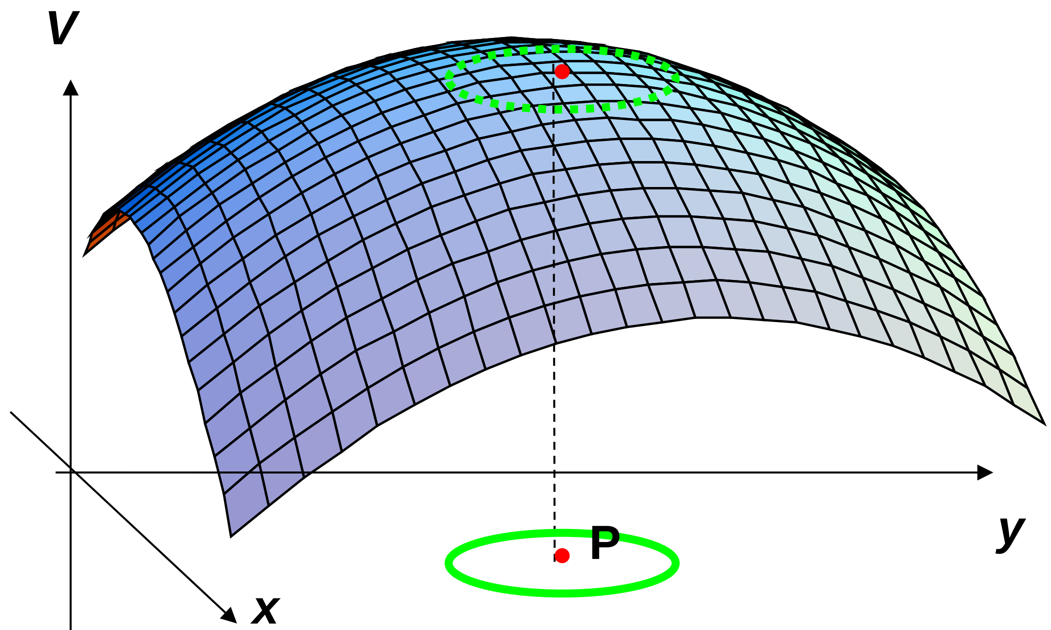
$$V(\mathbf{r}) < V(\mathbf{P})$$

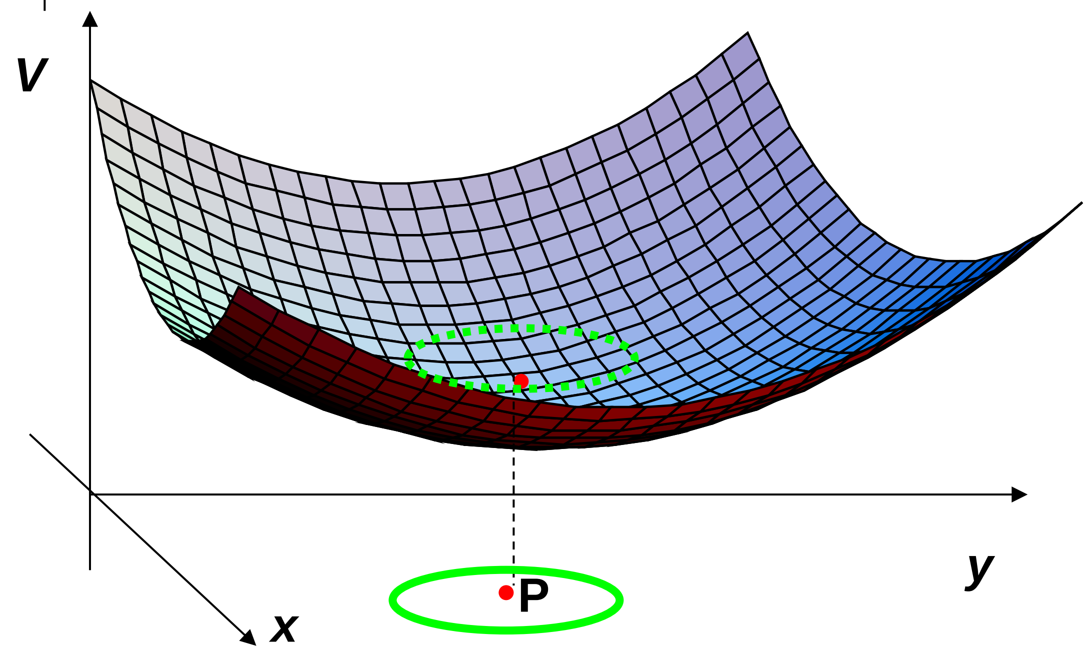
or

$$V(\mathbf{r}) > V(\mathbf{P})$$

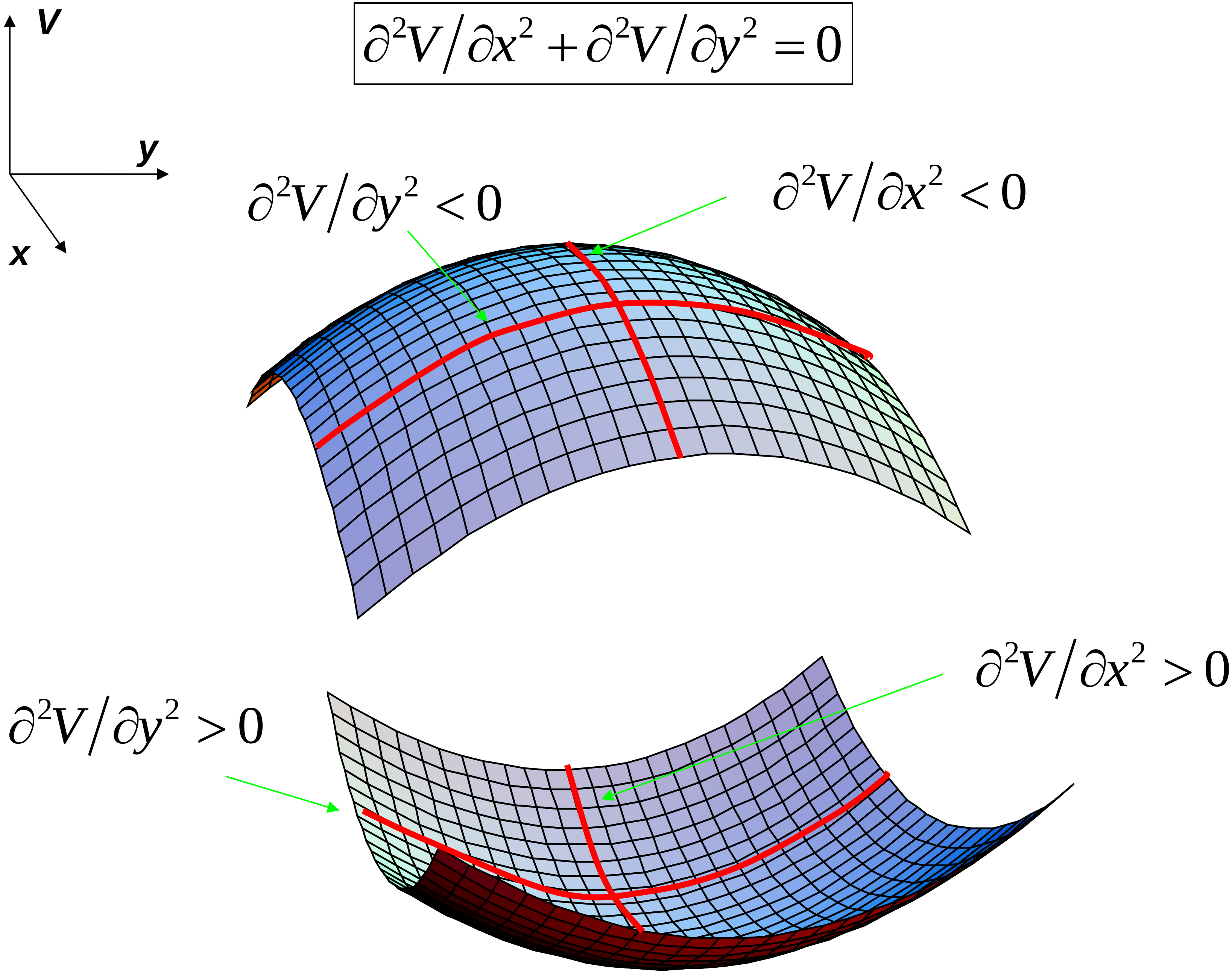
- In either case,

$$\langle V \rangle_S \neq V(\mathbf{P})$$

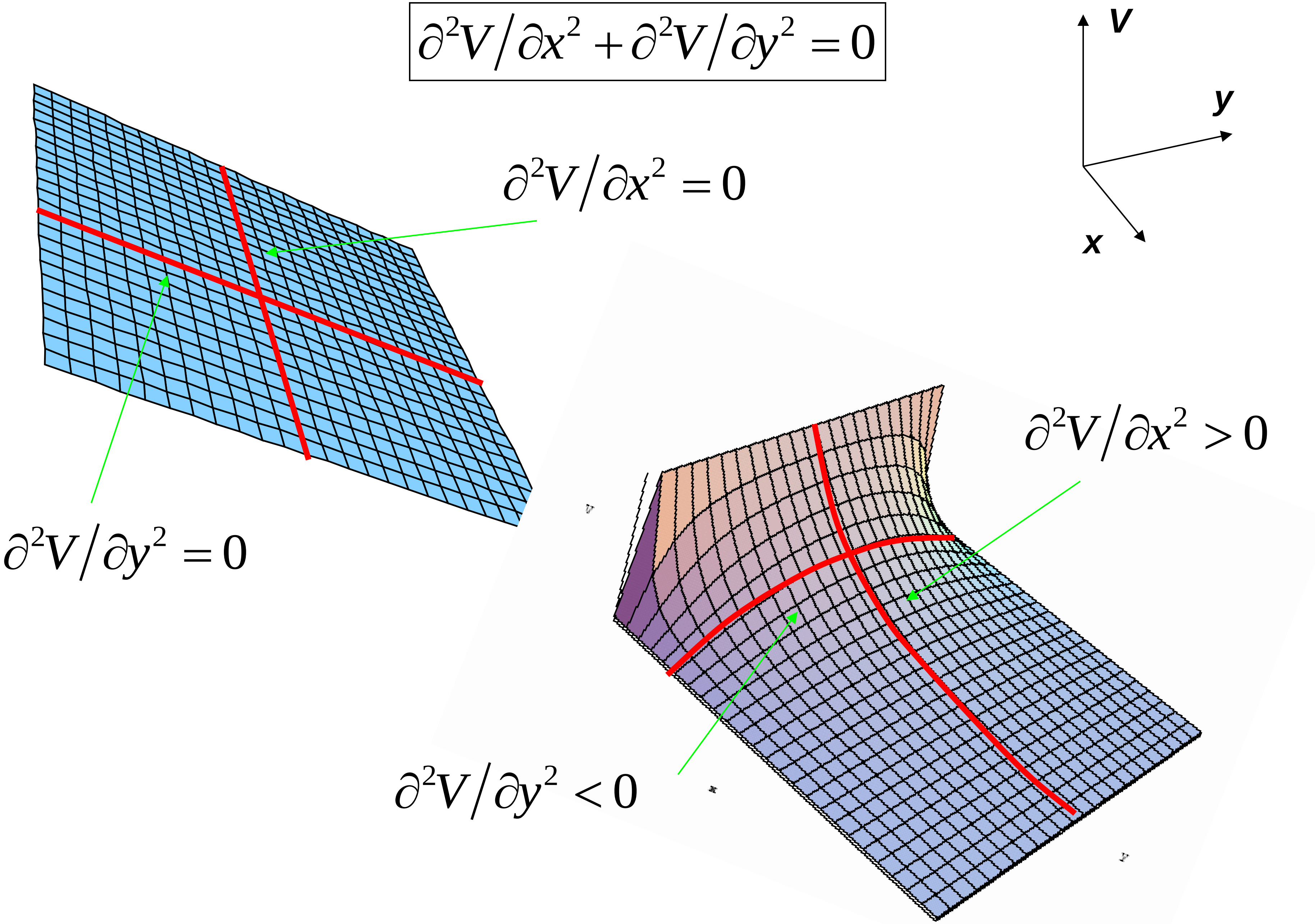




$$\boxed{\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 = 0}$$



$$\boxed{\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 = 0}$$

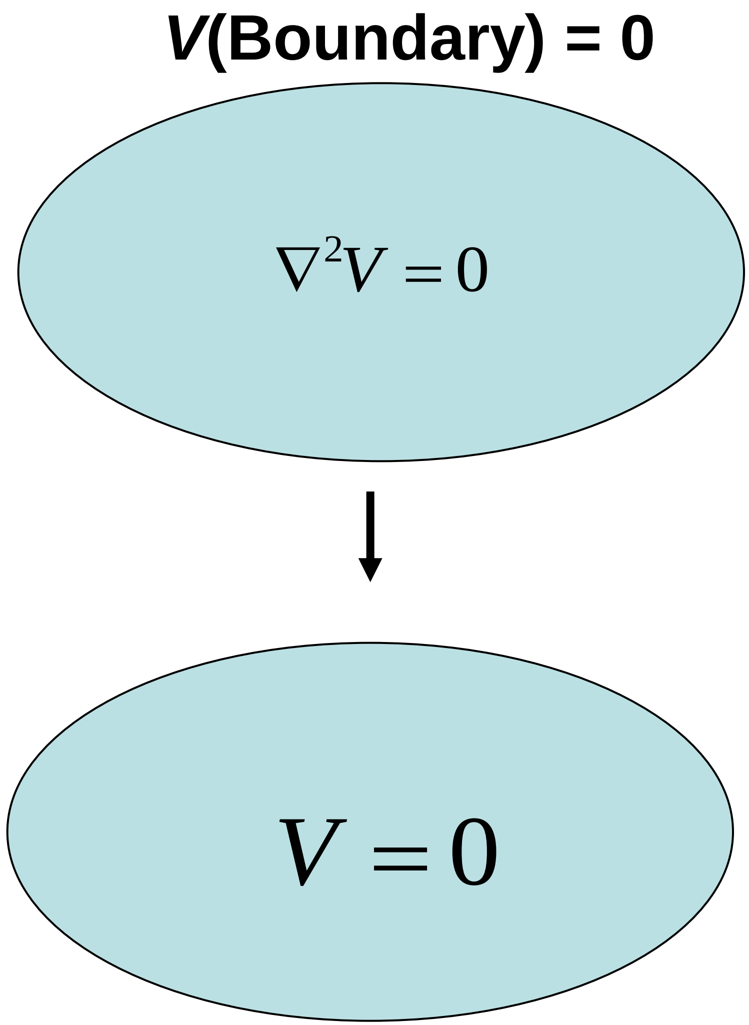


Properties of Harmonic Functions

3. By (2), it can be shown that if

- *V* satisfies the Laplace's equation inside a region
- *V* = 0 on the boundary

then *V* = 0 at all points inside the region



Proof

- V has no local maximum or minimum inside
- Because V(Boundary) = 0
- hence $V = 0$ inside the region

Existence and Uniqueness Theorem of Poisson’s Equation and Laplace’s Equation

Existence Theorem

- It can be shown that for the Poisson's equation (and in particular, the Laplace's equation) inside a region, *when suitable conditions are imposed on the boundary of the region, solutions do exist.*

Uniqueness Theorem

- It can be shown that for the Poisson's equation (and in particular, the Laplace's equation) inside a region, when suitable conditions are imposed on the boundary of the region, *if a solution exists, then it is the unique solution*
- This ensures that once we have constructed a solution which satisfies both the differential equation and the given boundary conditions, we do not need to look for other possible solutions

Uniqueness Theorem

The solution to Possion’s (Laplace’s) equation in some region is uniquely determined if the value of *V* is specified on all boundaries of the region

Proof:

Suppose there are two solutions V_1 and V_2 to the Poisson's equation satisfying the same boundary conditions, i.e.,

$$\nabla^2 V_1 = -\frac{\rho}{\varepsilon_0}, \qquad \nabla^2 V_2 = -\frac{\rho}{\varepsilon_0}$$

$$V_1\left(\text{Boundary}\right) = V_2\left(\text{Boundary}\right)$$

Consider $$V_3 = V_1 - V_2$$

We have $$\nabla^2 V_3 = 0$$

$$V_3\left(\text{Boundary}\right) = 0$$

Therefore, $$V_3 = 0$$

In other words, $$V_1 = V_2$$

In particular, the uniqueness theorem holds for V satisfying the Laplace's equation.

Solutions of Poisson's Equation

- Most of the systems we encounter in EM are of this form:
  - We are only interested in obtaining the fields inside a certain region
  - The source distribution inside the region is known
  - The source distributions on the boundaries may not be known, but given implicitly by boundary conditions of the potentials in certain form
- Hence the problem is reduced to solving the Poisson’s equation with given boundary conditions

- For E field: $\nabla^2 V = -\rho/\varepsilon_0$
- We shall learn how to construct a solution satisfying the differential equation and the given boundary condition.
- Because of the uniqueness theorem, once we have constructed one solution, we know that it is the only solution to the problem.

- The general strategy of solving the Poisson’s equation with given boundary conditions is beyond the scope of this course. We shall only consider simpler cases when:

1. the region $\mathcal{V}$ is the entire space, the source is localized, and the boundary condition is that the potential vanishes at infinity.
2. the source distribution and boundary conditions are of particular forms which allow one to construct the solution easily by using simple “image” sources.
3. the source distribution is of particular forms which allow one to reduce the Poisson equation to the simpler Laplace’s equation.

Solution For (1)

- $\nabla^2 V = -\rho / \varepsilon_0$
- the source ρ is localized
- $V(\infty) = 0$

$$\longrightarrow \quad V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_{\text{entire space}} \frac{\rho(\mathbf{r}')}{\mathscr{r}} d\tau'$$

Proof

$$\nabla_{\mathbf{r}}^2 V(\mathbf{r}) = \nabla_{\mathbf{r}}^2 \frac{1}{4\pi\varepsilon_0} \int_{\text{entire space}} \frac{\rho(\mathbf{r}')}{\mathscr{r}} d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0} \int_{\text{entire space}} \rho(\mathbf{r}') \nabla_{\mathbf{r}}^2 \left(\frac{1}{\mathscr{r}} \right) d\tau'$$

$$\nabla_{\mathbf{r}}^{2}\frac{1}{r}=\nabla_{\mathbf{r}}\cdot\left(-\frac{\hat{\mathbf{r}}}{r^{2}}\right)=-\nabla_{\mathbf{r}}\cdot\left(\frac{\hat{\mathbf{r}}}{r^{2}}\right)=-4\pi\delta^{3}\left(r\right)=-4\pi\delta^{3}\left(\mathbf{r}-\mathbf{r}'\right)$$

$$\nabla_{\mathbf{r}}^{2}V\left(\mathbf{r}\right)=\frac{1}{4\pi\varepsilon_{0}}\int_{\text{entire space}}\rho\left(\mathbf{r}'\right)\left[-4\pi\delta^{3}\left(\mathbf{r}-\mathbf{r}'\right)\right]d\tau'$$

$$=-\frac{1}{\varepsilon_{0}}\int_{\text{entire space}}\rho\left(\mathbf{r}'\right)\delta^{3}\left(\mathbf{r}-\mathbf{r}'\right)d\tau'$$

$$=-\rho\left(\mathbf{r}\right)/\varepsilon_{0}$$

- → It satisfies the Poisson’s equation

- When ρ is localized, it is obvious that

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_{\text{entire space}} \frac{\rho(\mathbf{r}')}{\mathscr{r}} d\tau' \to 0$$

when $r \to \infty$

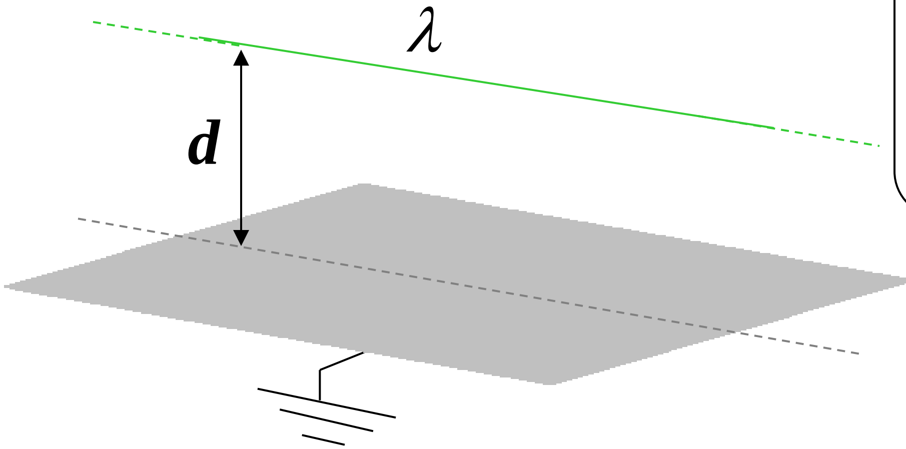
- It satisfies the boundary condition as well
- It is the unique solution by the first uniqueness theorem

Special Techniques for Calculating Potentials

The Method of Images

The Method of Images:

Suppose we want to find the solution of the Poisson's equation in a region $\mathcal{V}$ with specified boundary conditions. e.g.



What is the potential above the infinite plane?

The Method of Images:

The original problem may be difficult to solve.

We can try to add some image charges ***outside*** $\mathcal{V}$ *, such that the boundary conditions are fulfilled.*

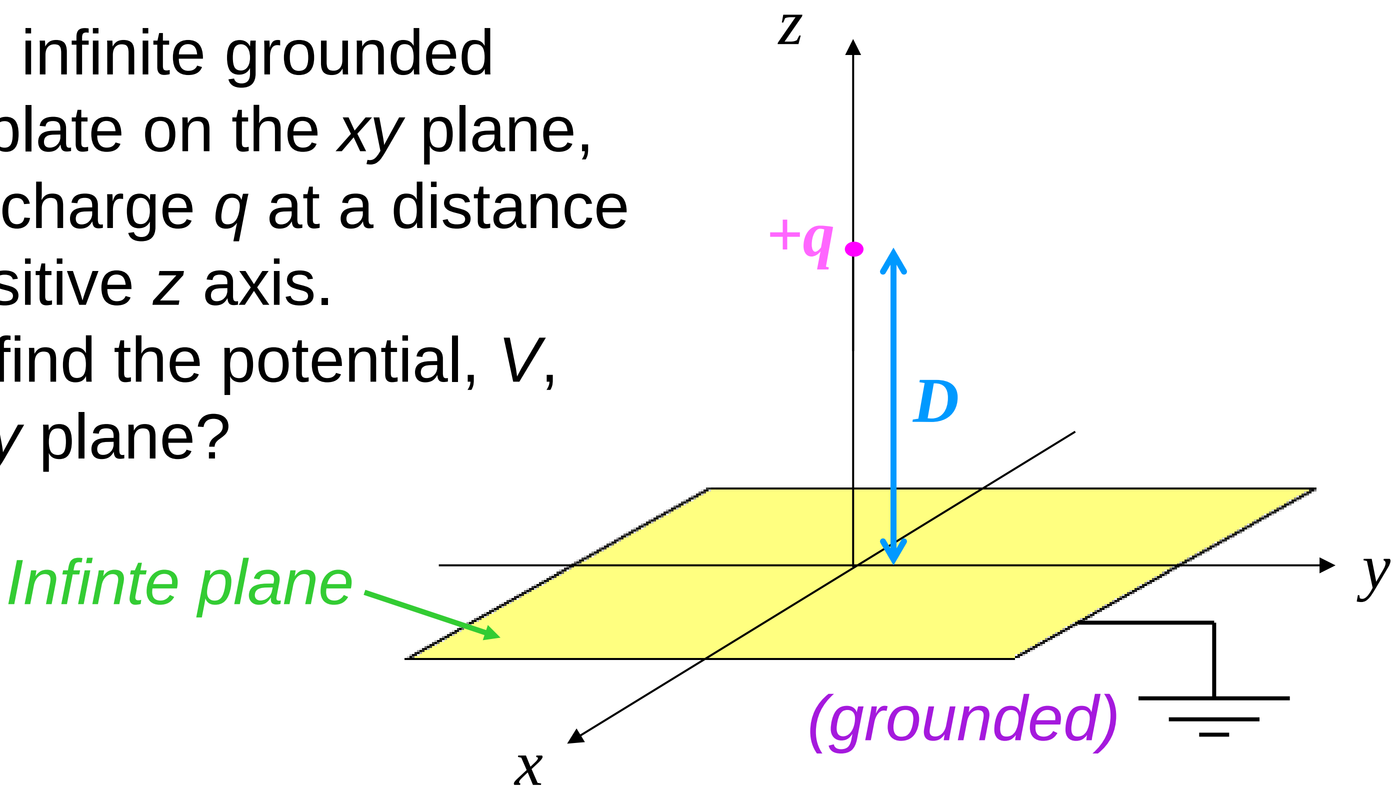
The Method of Images:

- Since the charges in $\mathcal{V}$ are not changed, the potential inside $\mathcal{V}$ satisfies the original Poisson's equation
- Hence by the uniqueness theorem, the potential in $\mathcal{V}$ is the solution of the original problem
- Notice that by adding image charges, the Poisson's equation outside $\mathcal{V}$ is changed, and our solution does not give correctly the potential outside

The Classic Image Problem:

Consider an infinite grounded conducting plate on the *xy* plane, with a point charge *q* at a distance *D* on the positive *z* axis.

How do we find the potential, *V*, above the *xy* plane?



V satisfies the Poisson's equation

$$\nabla^2 V = -q\delta\left(\mathbf{r} - \mathbf{r}_0\right) / \varepsilon_0$$

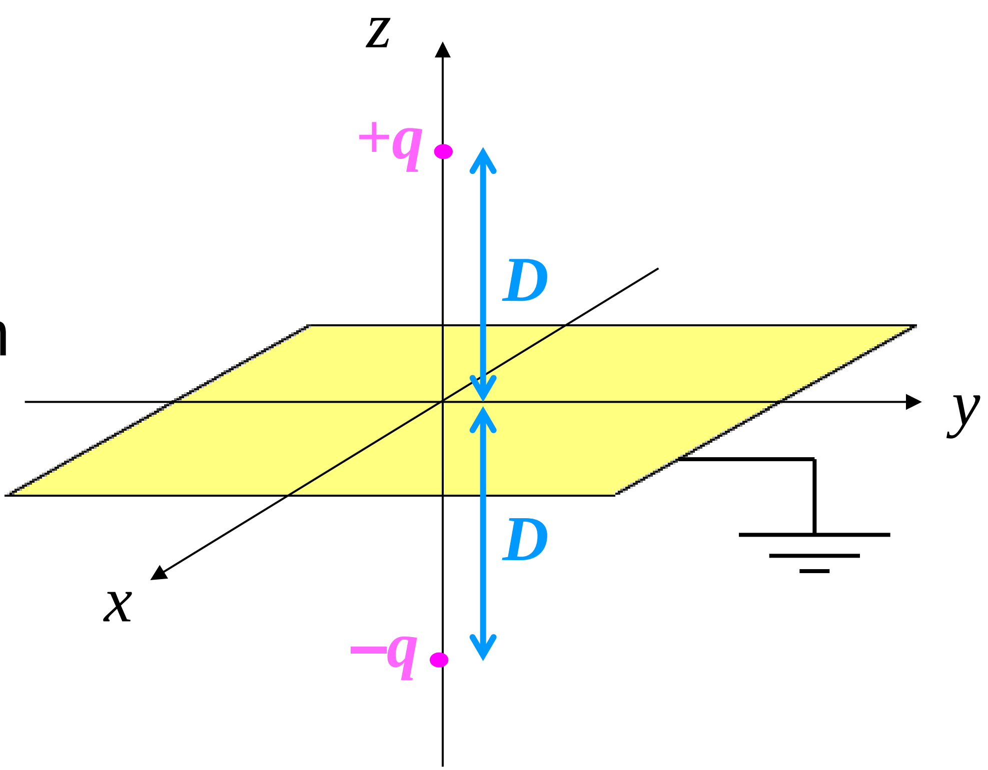
above the xy plane, where $\mathbf{r}_0$ is the position of the point charge.

The boundary condition (B.C.) is

1) $V = 0$ on the xy plane (*∵grounded*)
2) $V \rightarrow 0$ when $x^2 + y^2 + z^2 \gg D^2$

One cannot solve the problem by the direct method of integration with the Coulomb’s law (since there is induced charge distribution on the conducting plate, which is unknown)

Instead, consider the configuration without the conducting plate, but with another charge –*q* at $-D\hat{\mathbf{z}}$



The potential of this system can readily be obtained as

$$V(x,y,z)=\frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-D)^2}}-\frac{1}{\sqrt{x^2+y^2+(z+D)^2}}\right]$$

$$V(x,y,z)=\frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-D)^2}}-\frac{1}{\sqrt{x^2+y^2+(z+D)^2}}\right]$$

It is obvious that

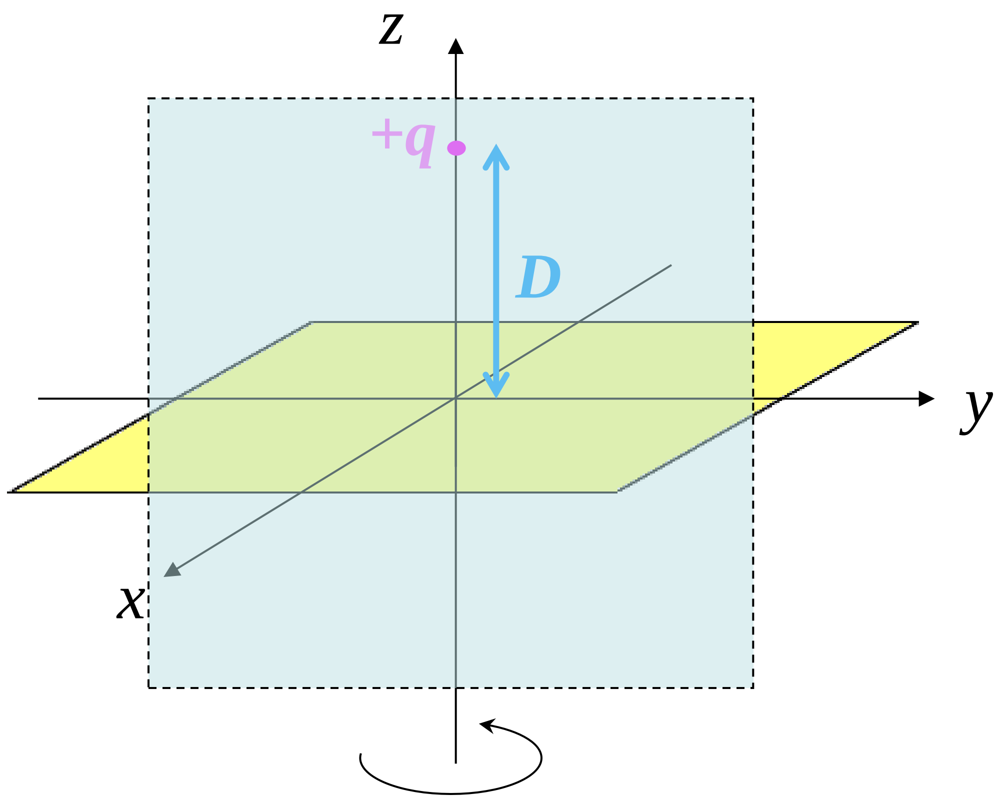
(1) $\underbrace{\nabla^2 V=-q\delta(\mathbf{r}-D\hat{\mathbf{z}})/\varepsilon_0}_{V\text{ satisfies the Poisson's equation}}$ above the xy plane,

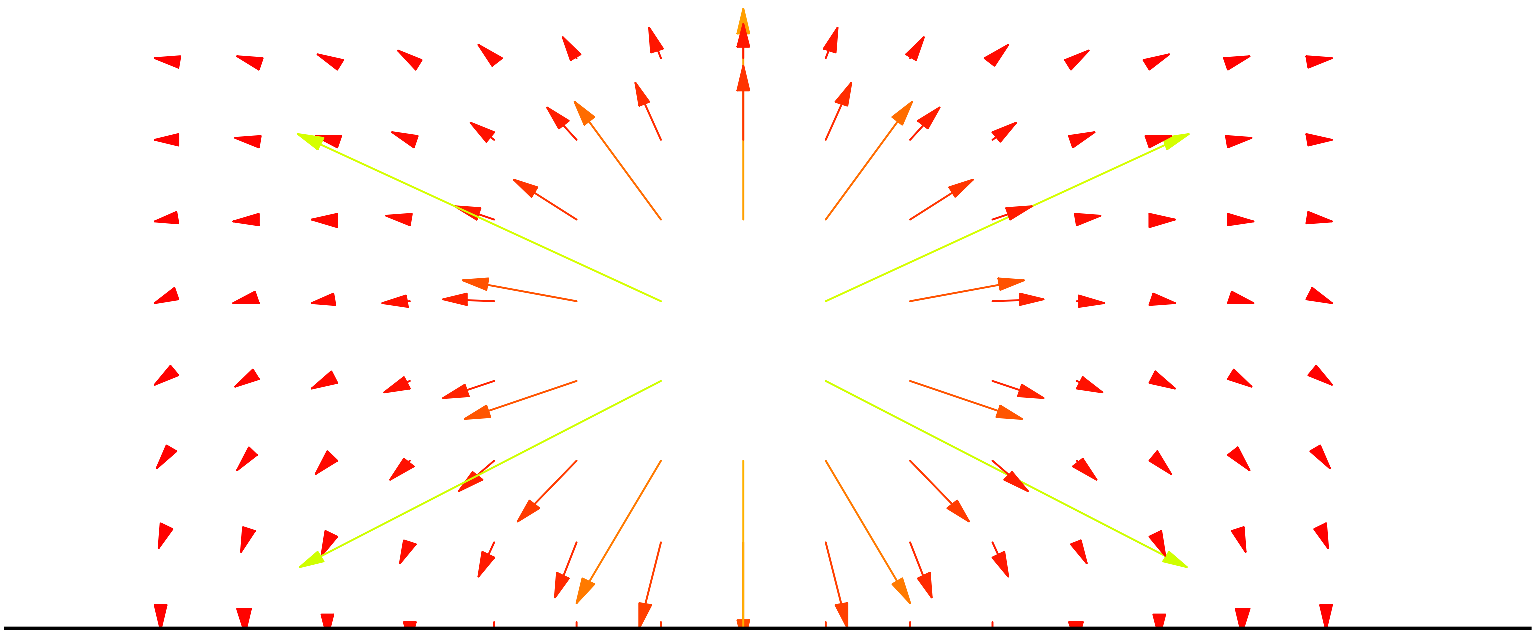
(2) $V(x,y,0)=0$, and

(3) $V(x,y,z)\to 0$, when $x^2+y^2+z^2 \gg D^2$

$\left.\begin{matrix}(2)\\(3)\end{matrix}\right\}$ *V satisfies B.C.*

By the uniqueness theorem, V above the xy plane is the potential of the original problem with the conducting plate.

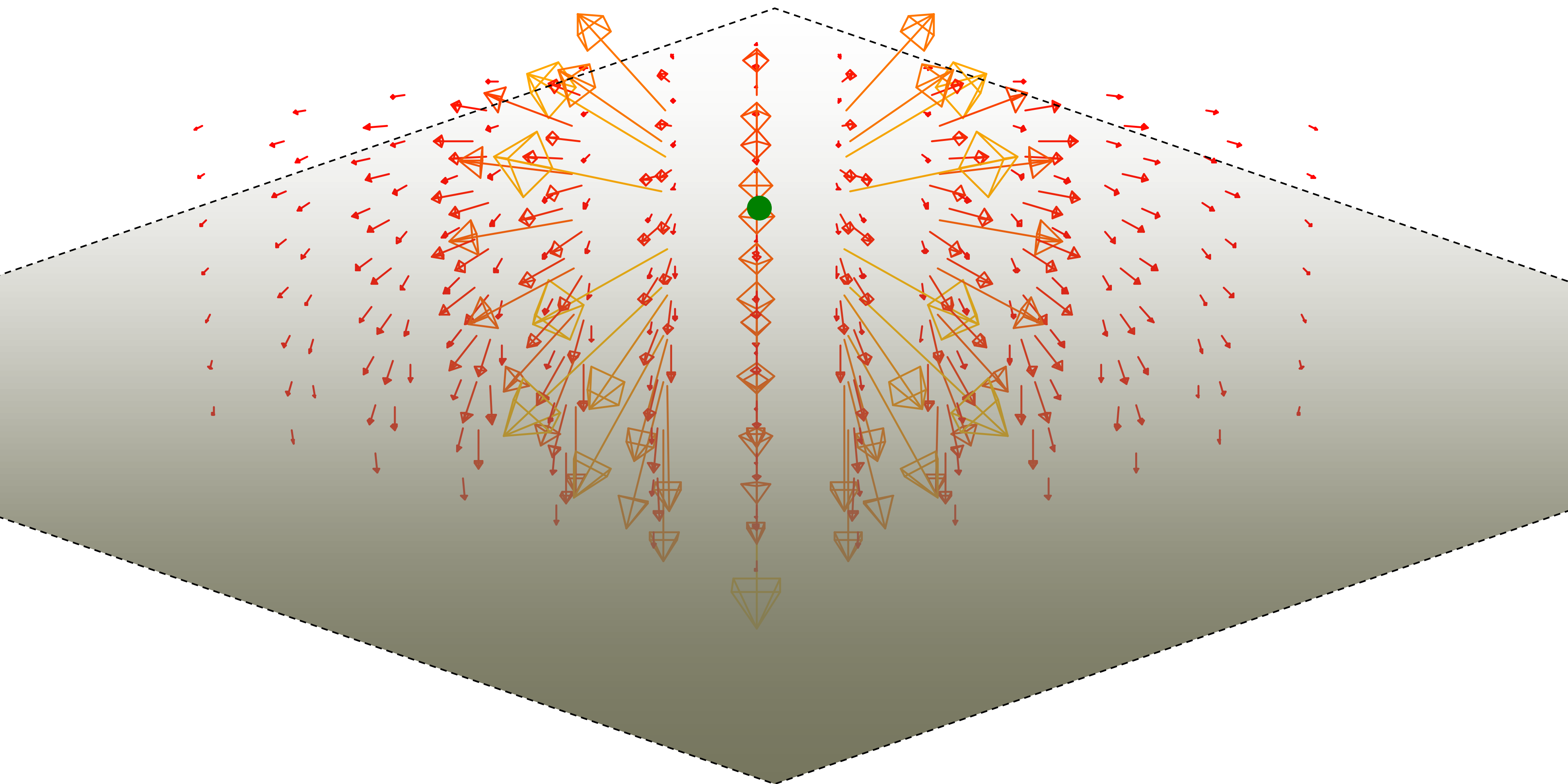

z
+q
D
y
x



$$\mathbf{E}(x,y,z)=\frac{q}{4\pi\varepsilon_0}\left(\frac{x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+(z-D)\hat{\mathbf{z}}}{\left(x^2+y^2+(z-D)^2\right)^{3/2}}-\frac{x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+(z+D)\hat{\mathbf{z}}}{\left(x^2+y^2+(z+D)^2\right)^{3/2}}\right)$$

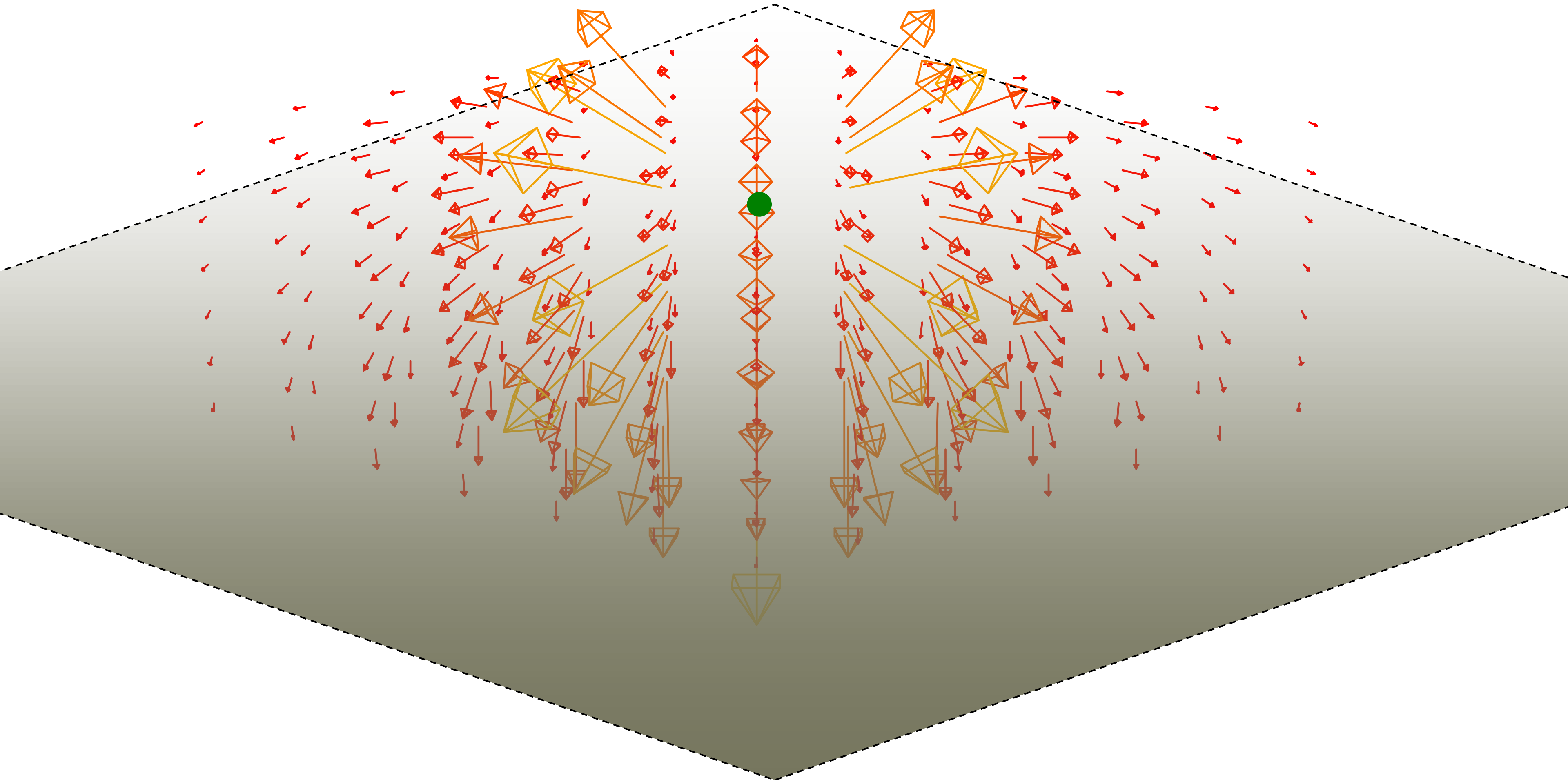
$$V(x,y,z)=\frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-D)^2}}-\frac{1}{\sqrt{x^2+y^2+(z+D)^2}}\right]$$

$$\mathbf{E}(x,y,z)=\frac{q}{4\pi\varepsilon_0}\left(\frac{x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+(z-D)\hat{\mathbf{z}}}{\left(x^2+y^2+(z-D)^2\right)^{3/2}}-\frac{x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+(z+D)\hat{\mathbf{z}}}{\left(x^2+y^2+(z+D)^2\right)^{3/2}}\right)$$



Example

- Find the surface charge density and the total amount of charge induced on the plate



Answer

- Above the xy plane:

$$\mathbf{E}(x,y,z)=\frac{q}{4\pi\varepsilon_0}\left(\frac{x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+(z-D)\hat{\mathbf{z}}}{\left(x^2+y^2+(z-D)^2\right)^{3/2}}-\frac{x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+(z+D)\hat{\mathbf{z}}}{\left(x^2+y^2+(z+D)^2\right)^{3/2}}\right)$$

- Just above the xy plane:

$$\mathbf{E}(x,y,z)=\frac{q}{4\pi\varepsilon_0}\left(\frac{x\hat{\mathbf{x}}+y\hat{\mathbf{y}}-D\hat{\mathbf{z}}}{\left(x^2+y^2+D^2\right)^{3/2}}-\frac{x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+D\hat{\mathbf{z}}}{\left(x^2+y^2+D^2\right)^{3/2}}\right)$$

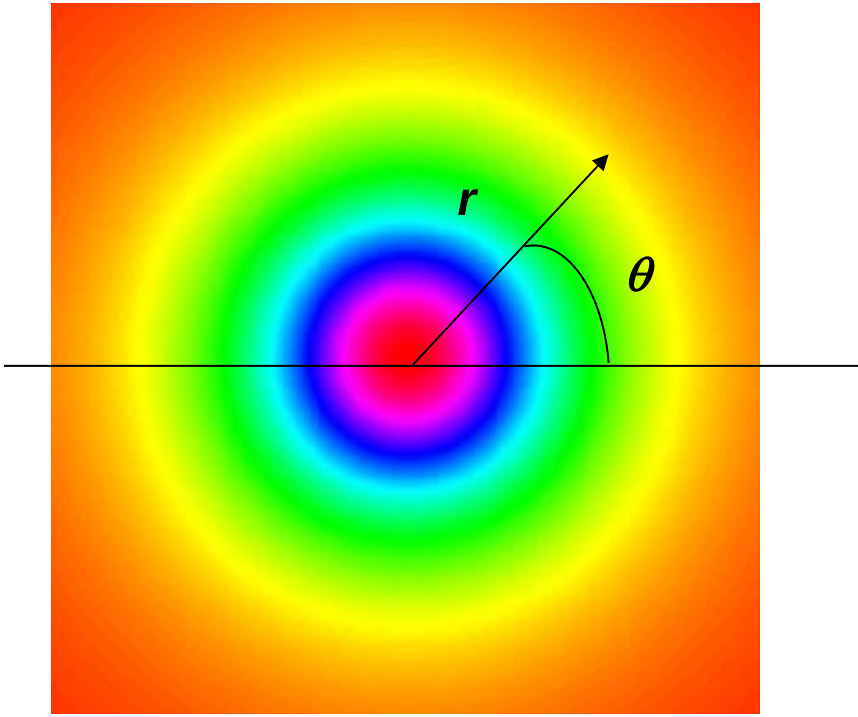
$$=-\frac{qD}{2\pi\varepsilon_0\left(x^2+y^2+D^2\right)^{3/2}}\hat{\mathbf{z}}$$

- However, we know that $\mathbf{E} = \dfrac{\sigma(x,y)}{\varepsilon_0}\hat{\mathbf{z}}$
- Hence

$$\sigma(x,y) = -\frac{qD}{2\pi\left(x^2+y^2+D^2\right)^{3/2}}$$

- In polar coordinates of the *xy* plane: $\sigma(r,\theta) = -\dfrac{qD}{2\pi\left(r^2+D^2\right)^{3/2}}$
- Hence the total induced charge is

$$\begin{aligned}
Q &= \int \sigma da \\
&= \int_{\theta=0}^{2\pi}\int_{r=0}^{\infty} -\frac{qD}{2\pi\left(r^2+D^2\right)^{3/2}} r dr d\theta \\
&= -\frac{qD}{2\pi}\int_{\theta=0}^{2\pi} d\theta \int_{r=0}^{\infty} \frac{rdr}{\left(r^2+D^2\right)^{3/2}} \\
&= -qD\left[-\frac{1}{\sqrt{r^2+D^2}}\right]_0^{\infty} \\
&= -q
\end{aligned}$$



Special Techniques for Calculating Potentials

Separation of Variables

What is Separation of Variable?

- Look for solutions that are products of functions, each of which depends on only one of the coordinates
- *e.g. the solution can be written in this form*

$$A(x,y)=\underbrace{B(x)}_{\text{function of } x}\underbrace{C(y)}_{\text{function of } y}$$

Cartesian Coordinates

In 2D case,

Laplace's equation: $$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0$$

Look for solutions of the form

$$V(x,y)=X(x)Y(y)$$

Sub. $V(x,y) = X(x)Y(y)$ into the Laplace's equation,
We have

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$$

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$$

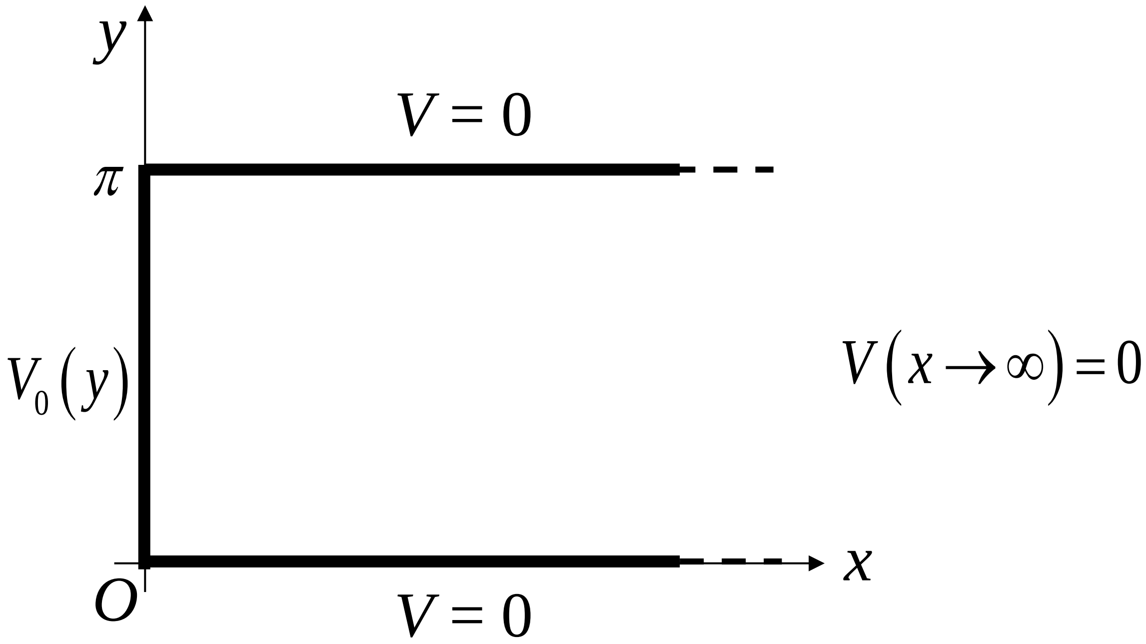
$$\Rightarrow \begin{cases} \dfrac{1}{X}\dfrac{d^2X}{dx^2} = \lambda \\ \dfrac{1}{Y}\dfrac{d^2Y}{dy^2} = -\lambda \end{cases}$$

,where λ is a constant.

whether one should choose λ to be positive or negative depends on the boundary conditions.

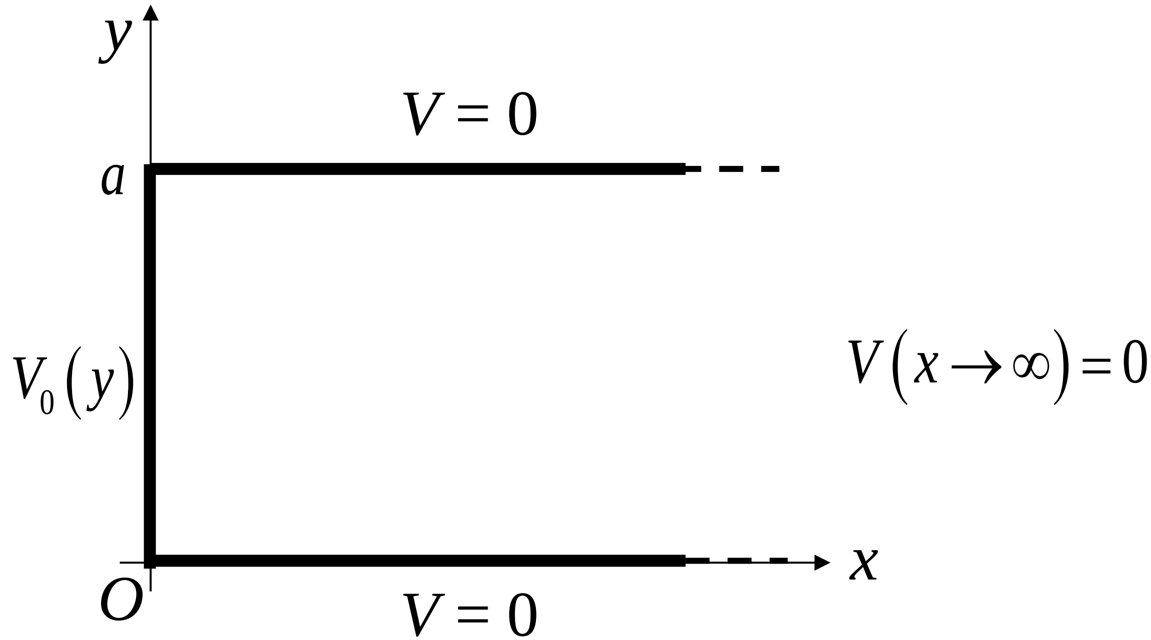
Example:

Find the potential inside the region $x > 0,\ 0 < y < \pi$, with the B.C. given as shown in the above figure. The potential goes to zero at the right hand side when $x \to \infty$



Solution:

$$\begin{cases} \dfrac{1}{X}\dfrac{d^2X}{dx^2} = \lambda \\ \dfrac{1}{Y}\dfrac{d^2Y}{dy^2} = -\lambda \end{cases}$$



The constant λ determines whether we have oscillatory or exponential solutions

If $\lambda < 0$, then X is oscillatory while Y is exponentially growing or decaying

In this case, we cannot have $V \to 0$ as $x \to \infty$

Therefore, one must choose $\lambda > 0$, i.e.,

$$\begin{cases} \dfrac{d^2X}{dx^2} = k^2 X \\ \dfrac{d^2Y}{dy^2} = -k^2 Y \end{cases}$$,where k is real.

How about k ?

$$\begin{cases} \dfrac{d^2 X}{dx^2} = k^2 X \\ \dfrac{d^2 Y}{dy^2} = -k^2 Y \end{cases}$$

If $k = 0$, then

$$\begin{cases} X = Ax + B \\ Y = Cy + D \end{cases}$$

In this case, again we cannot have $V \to 0$ as $x \to \infty$

If $k \neq 0$, then

$$\begin{cases} X = Ae^{kx} + Be^{-kx} \\ Y = C \sin ky + D \cos ky \end{cases}$$

It can be observed that solutions for negative *k* are redundant if those for positive *k* are already taken into account. Hence we consider $k > 0$.

$$\boxed{V(x, y) = (Ae^{kx} + Be^{-kx})(C \sin ky + D \cos ky)}$$

Since $V \to 0$ as $x \to \infty$, we have

$$A = 0$$

In addition, $V = 0$ when $y = 0$ implies

$$D = 0$$

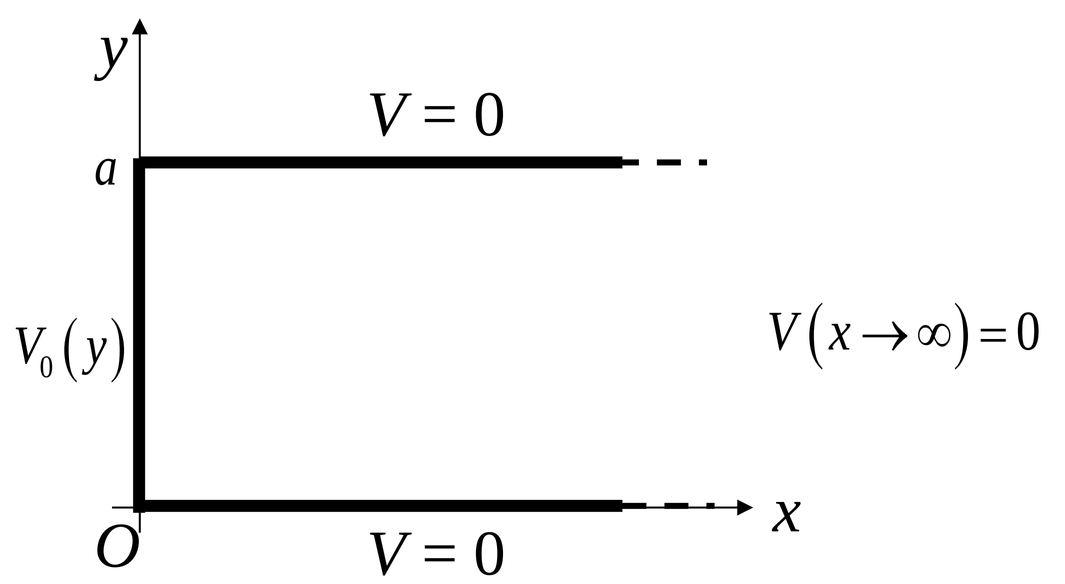
Hence
$$V = Ce^{-kx} \sin ky$$

The B.C. at the top plate, $V = 0$ when $y = a$, implies

$$\sin ka = 0$$

$$\Rightarrow k = \frac{n\pi}{a}, \quad (n = 1, 2, 3, \ldots)$$

$$\begin{cases} X = Ae^{kx} + Be^{-kx} \\ Y = C \sin ky + D \cos ky \end{cases}$$

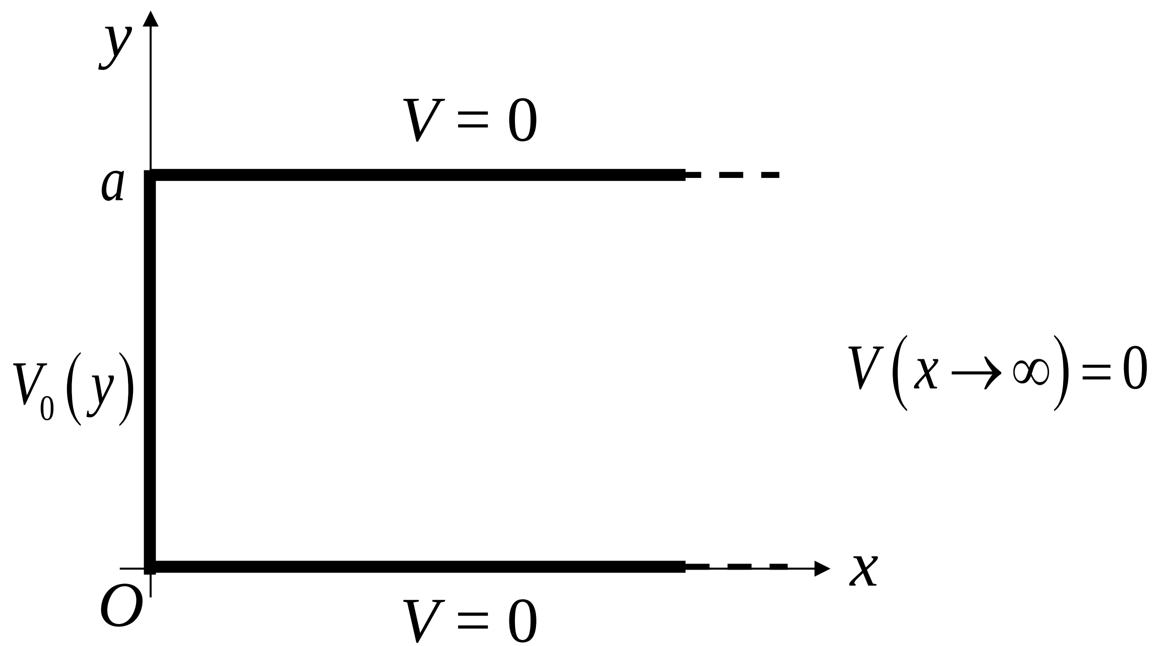


The solution:

$$V = Ce^{-kx}\sin ky$$

does not satisfy the B.C. at $x = 0$.

However, the ***Laplace's equation is linear***.
Therefore, any linear combination of the solutions is also a solution to the equation.



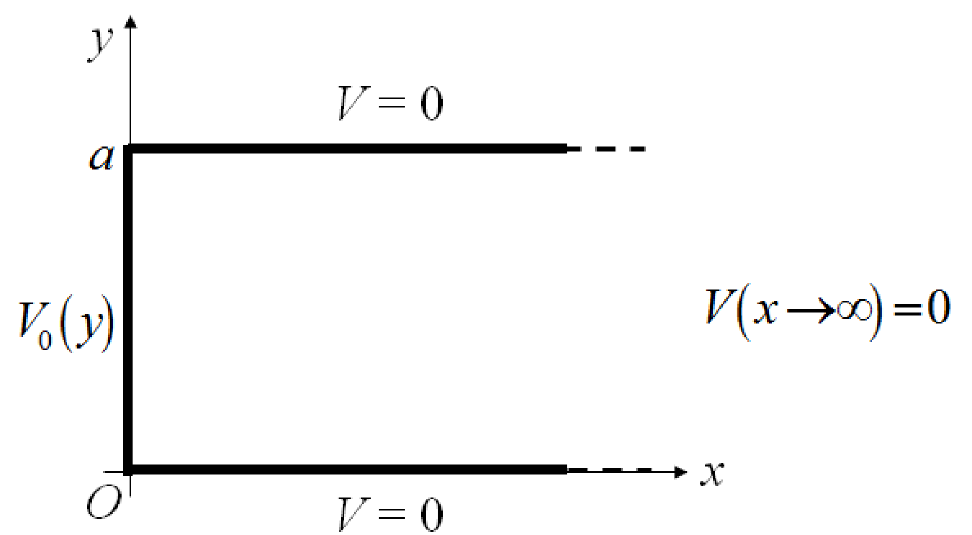
Therefore, We are looking for such a combination

$$\boxed{V(x,y)=\sum_{n=1}^{\infty} C_n e^{-n\pi x/a}\sin(n\pi y/a)}$$

so that the B.C. at $x = 0$ is satisfied.

In other words, we want to find C_k so that

$$V(0,y)=\sum_{n=1}^{\infty} C_n \sin(n\pi y/a) = V_0(y)$$



$$\sum_{k=1}^{\infty} C_k \sin ky = V_0\left(y\right)$$

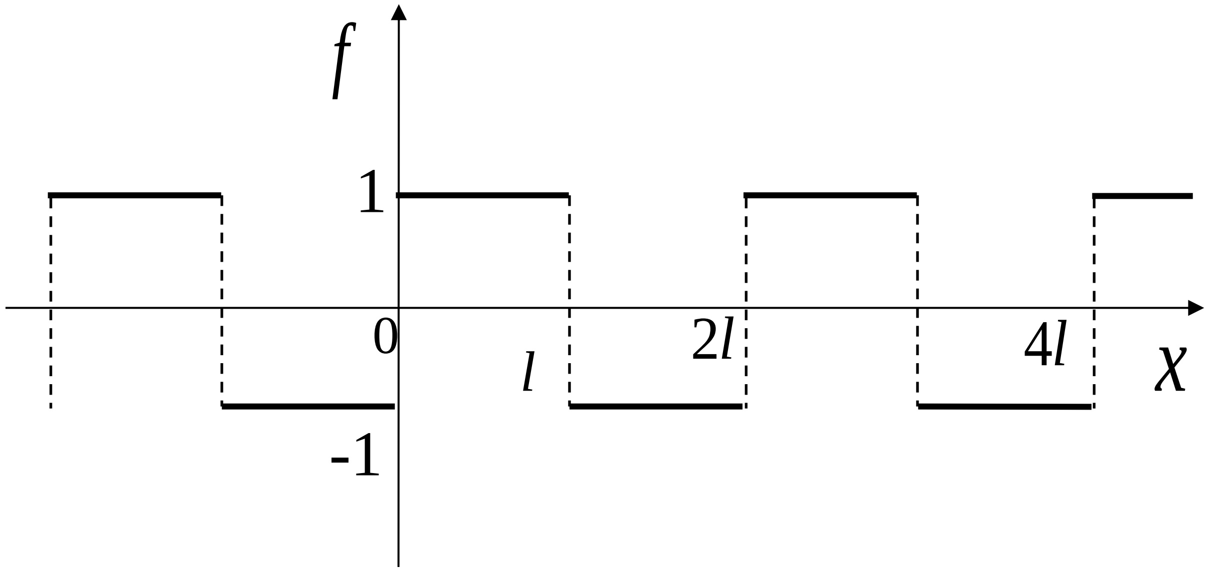
Is this always possible?

Yes!!

$$V(0, y) = \sum_{n=1}^{\infty} C_n \sin(n\pi y / a) = V_0\left(y\right)$$

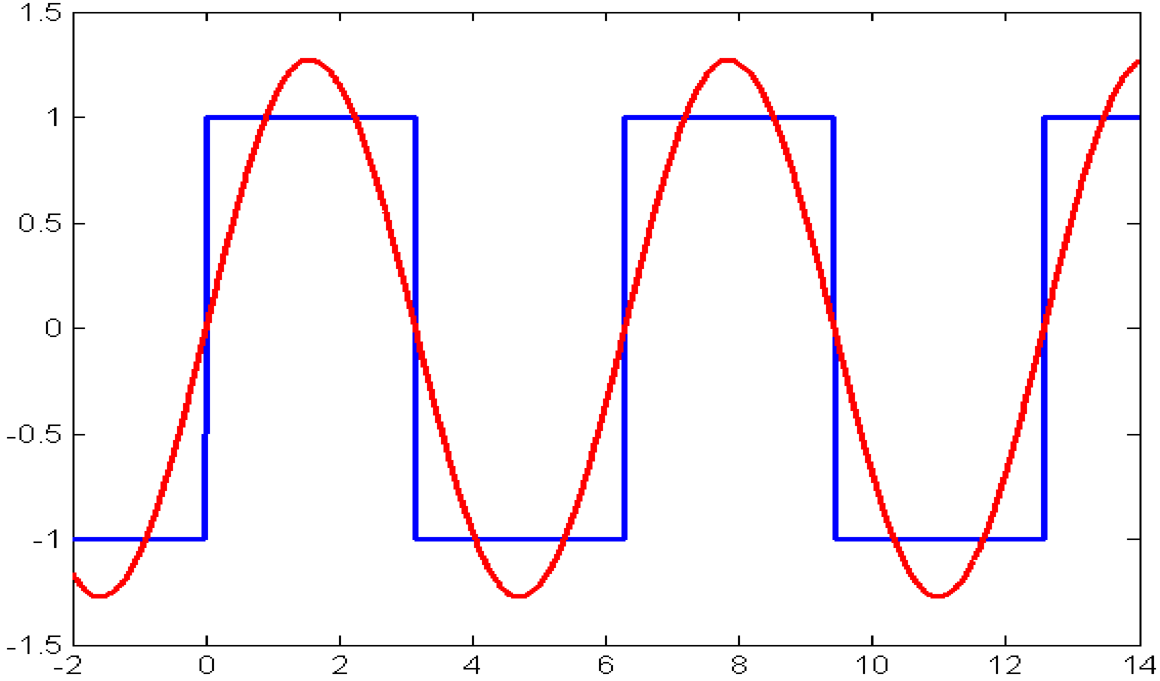
It is a Fourier series, any function $V_0(y)$ can be expanded in such a series

Example:
Fourier Synthesis of a Square Wave

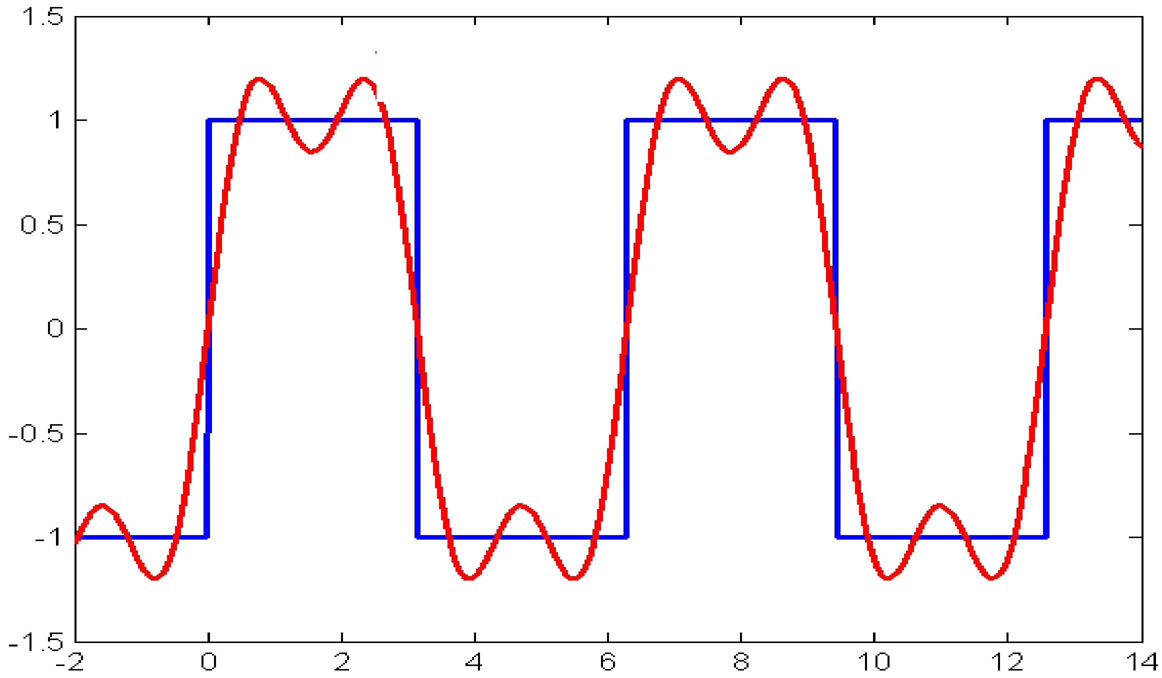


$$f(x) = \sum_{n=1,3,5\ldots}^{\infty} \frac{4}{n\pi} \sin \frac{n\pi x}{l}$$

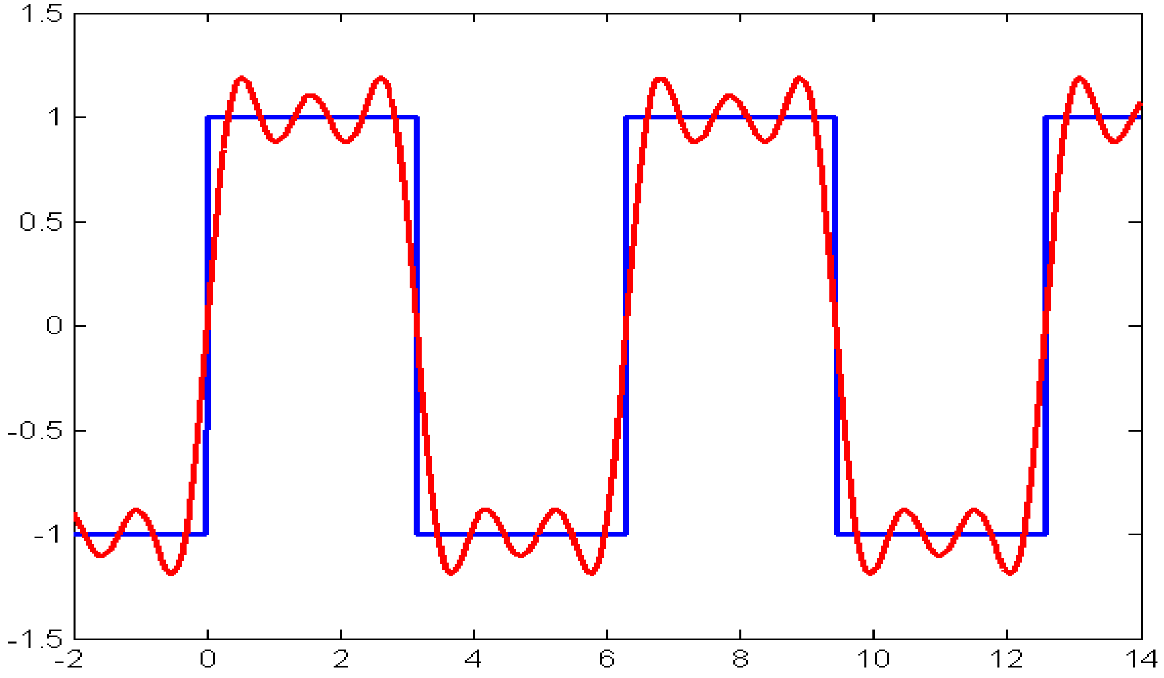
$$\frac{4}{\pi}\sin\omega_0 t$$



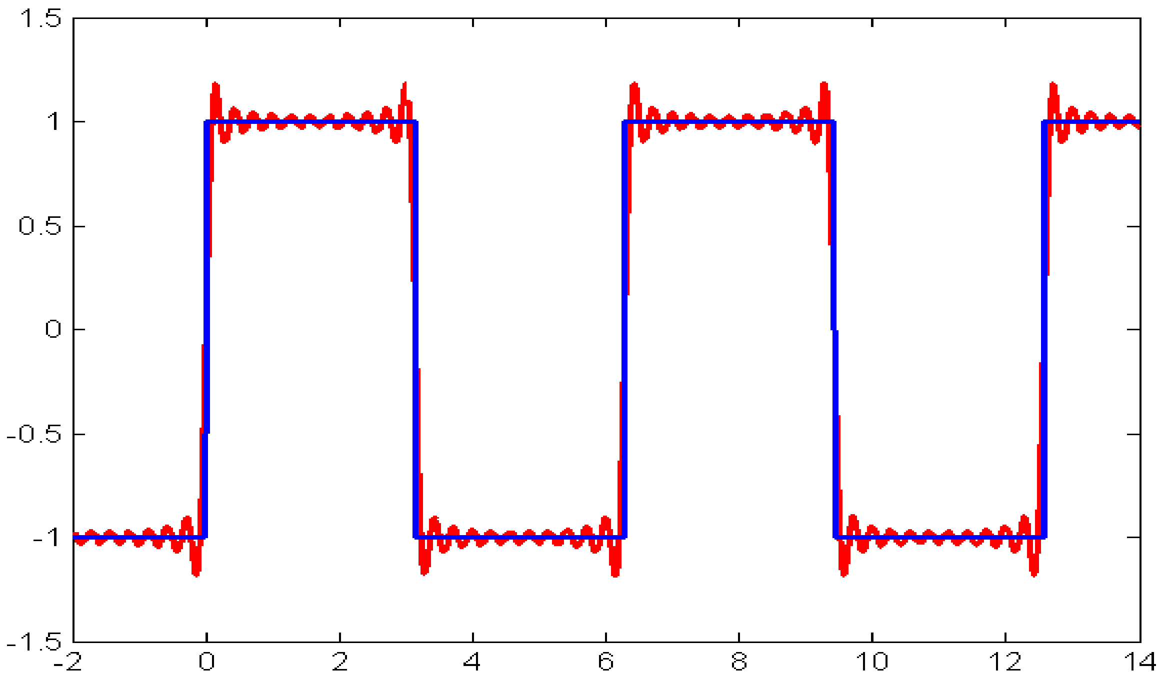
$$\frac{4}{\pi}\sin\omega_0 t+\frac{4}{3\pi}\sin 3\omega_0 t$$



$$\frac{4}{\pi}\sin\omega_0 t + \frac{4}{3\pi}\sin 3\omega_0 t + \frac{4}{5\pi}\sin 5\omega_0 t$$



$$\frac{4}{\pi}\sin\omega_0 t+\frac{4}{3\pi}\sin 3\omega_0 t+\cdots+\frac{4}{21\pi}\sin 21\omega_0 t+\frac{4}{23\pi}\sin 23\omega_0 t$$



$$V(0,y)=\sum_{n=1}^{\infty}C_n\sin(n\pi y/a)=V_0\left(y\right)$$

$$\sum_{n=1}^{\infty}C_n\int_0^a\sin(n\pi y/a)\,\sin(n'\pi y/a)\,dy=\int_0^a V_0(y)\sin(n'\pi y/a)\,dy$$

$$\int_0^a\sin(n\pi y/a)\,\sin(n'\pi y/a)\,dy=\begin{cases}0, & \text{if } n'\neq n,\\ \dfrac{a}{2}, & \text{if } n'=n.\end{cases}$$

Harmonic functions as solutions are orthogonal, only terms for *n*'=*n* remain!

$$C_n=\frac{2}{a}\int_0^a V_0(y)\sin(n\pi y/a)\,dy.$$

Solution:

$$V(x, y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin(n\pi y/a)$$

With the coefficients:

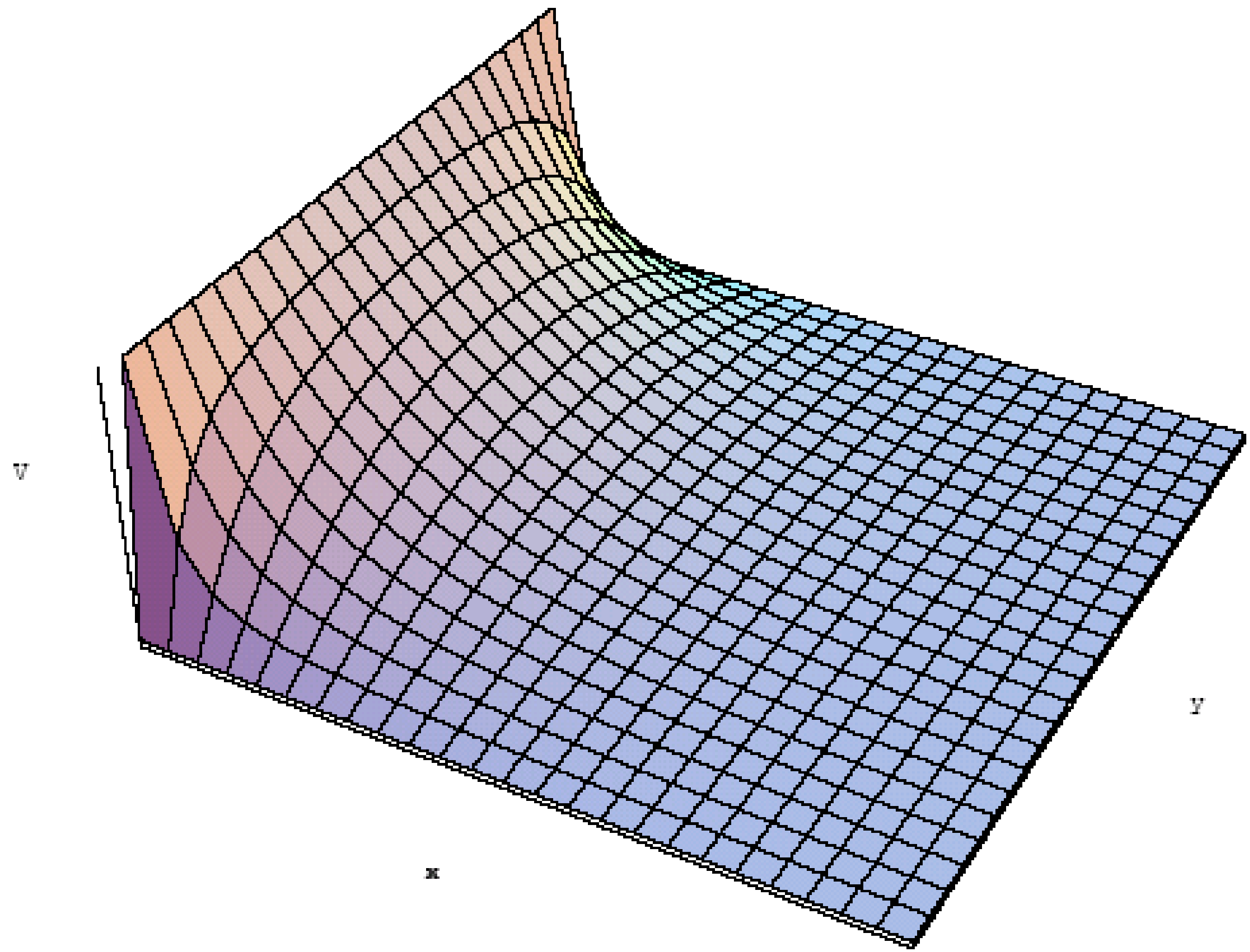
$$C_n = \frac{2}{a} \int_0^a V_0(y) \sin(n\pi y/a)\, dy.$$

If V_0 is a constant potential:

$$C_n = \frac{2V_0}{a}\int_0^a \sin(n\pi y/a)\,dy = \frac{2V_0}{n\pi}(1-\cos n\pi) = \begin{cases} 0, & \text{if } n \text{ is even,} \\ \dfrac{4V_0}{n\pi}, & \text{if } n \text{ is odd.} \end{cases}$$

$$V(x, y) = \frac{4V_0}{\pi}\sum_{n=1,3,5\ldots}\frac{1}{n}e^{-n\pi x/a}\sin(n\pi y/a).$$

The solution:



Cartesian Coordinates

In 3D case,

Laplace's equation: $$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0$$

Look for solutions of the form

$$V(x,y,z)=X(x)Y(y)Z(z)$$

Sub. $V(x,y,z) = X(x)Y(y)Z(z)$ into the Laplace's equation, we have

$$YZ\frac{d^2X}{dx^2} + XZ\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2} = 0$$

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0$$

$$\Rightarrow \begin{cases} \dfrac{1}{X}\dfrac{d^2X}{dx^2} = \lambda_1 \\ \dfrac{1}{Y}\dfrac{d^2Y}{dy^2} = \lambda_2 \\ \dfrac{1}{Z}\dfrac{d^2Z}{dz^2} = \lambda_3 \end{cases}$$

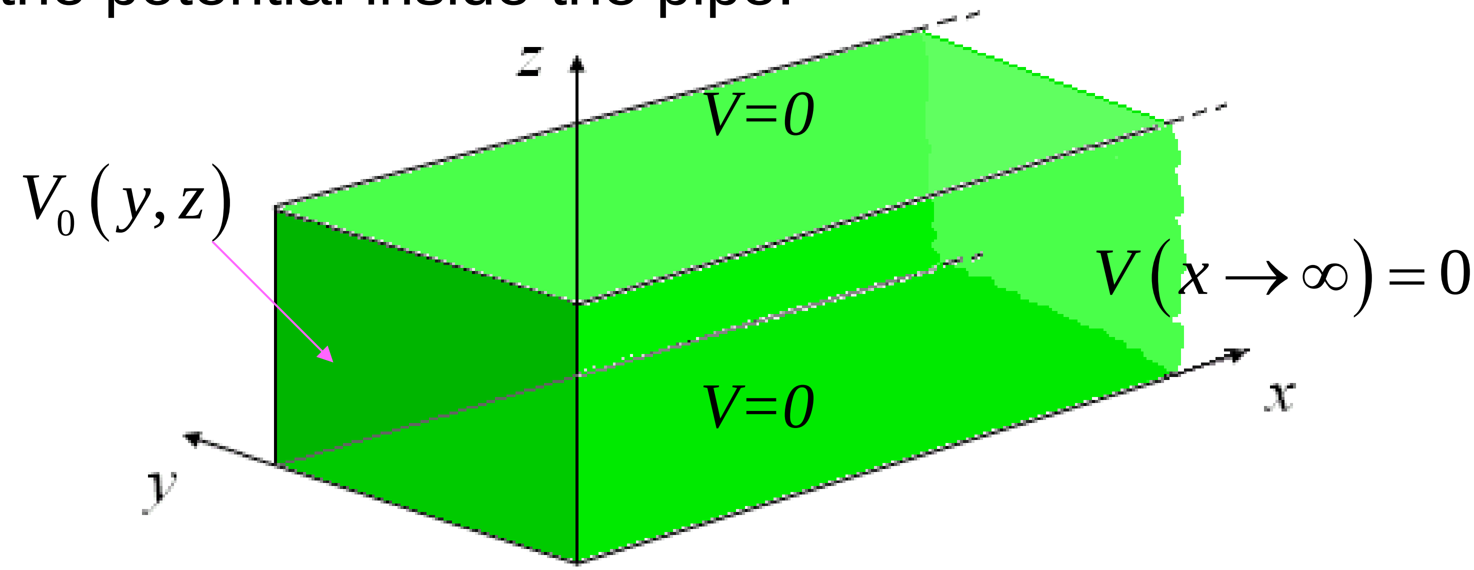
,where $\lambda_1 + \lambda_2 + \lambda_3 = 0$

Whether one should choose the constants to be ***positive*** *or* ***negative,*** *depends on the B.C.(s)*

Example:

An infinitely long square metal pipe (sides π) is grounded, but one end, at $x = 0$, is maintained at a specified potential $V_0(y,z)$, as shown in the figure below.

Find the potential inside the pipe.



$$\begin{cases} \dfrac{1}{X}\dfrac{d^2X}{dx^2} = \lambda_1 \\ \dfrac{1}{Y}\dfrac{d^2Y}{dy^2} = \lambda_2 \\ \dfrac{1}{Z}\dfrac{d^2Z}{dz^2} = \lambda_3 \end{cases}$$

In this case, we must choose $\lambda_1 > 0,\ \lambda_2, \lambda_3 < 0$

Let

$$\begin{cases} \dfrac{d^2X}{dx^2} = \left(k^2 + l^2\right) X \\ \dfrac{d^2Y}{dy^2} = -k^2 Y \\ \dfrac{d^2Z}{dz^2} = -l^2 Z \end{cases}$$

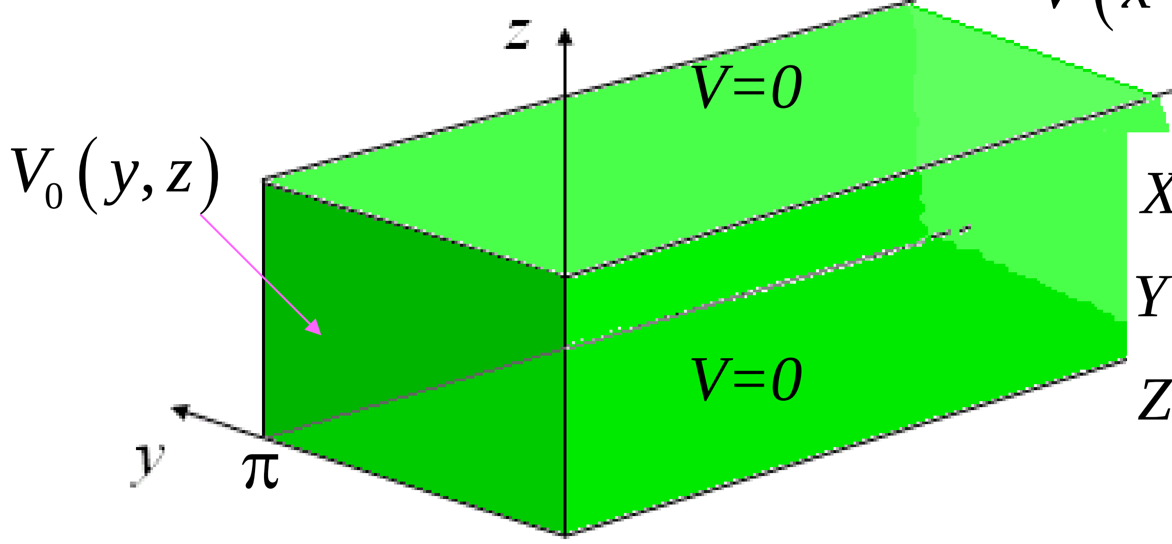
Then

$$X\left(x\right) = Ae^{\sqrt{k^2+l^2}\,x} + Be^{-\sqrt{k^2+l^2}\,x}$$

$$Y\left(y\right) = C\sin ky + D\cos ky$$

$$Z\left(z\right) = E\sin lz + F\cos lz$$

Again, we can assume $k, l > 0$



$$V(x \to \infty) = 0$$

$$X(x) = Ae^{\sqrt{k^2+l^2}x} + Be^{-\sqrt{k^2+l^2}x}$$

$$Y(y) = C\sin ky + D\cos ky$$

$$Z(z) = E\sin lz + F\cos lz$$

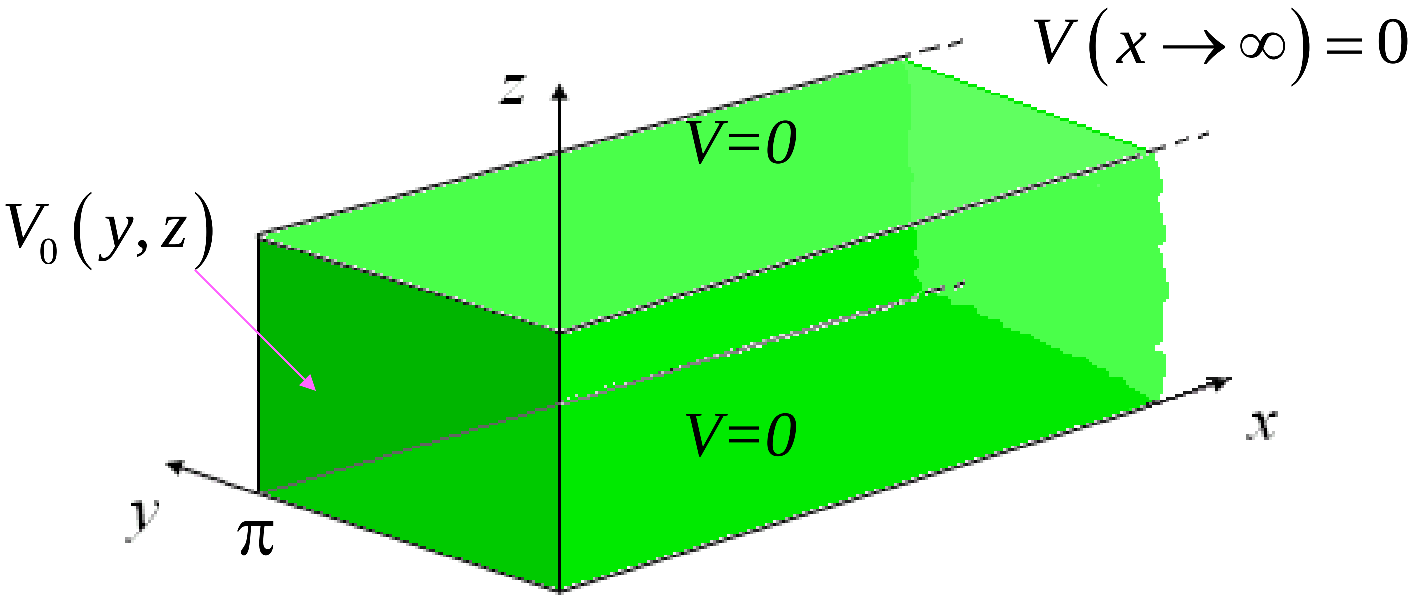
$V \to 0$ when $x \to \infty$ implies $A = 0$

$V = 0$ when $y = 0$ implies $D = 0$

$V = 0$ when $z = 0$ implies $F = 0$

Therefore,

$$V(x,y,z) = Ce^{-\sqrt{k^2+l^2}x}\sin ky \sin lz$$



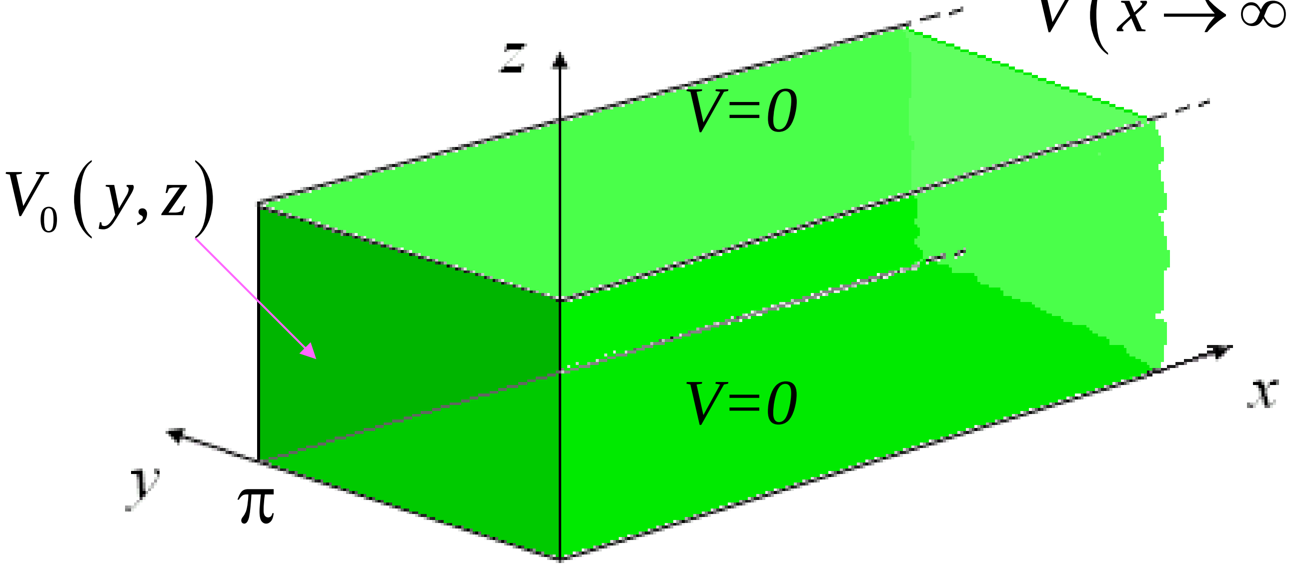
$$V(x,y,z) = Ce^{-\sqrt{k^2+l^2}x}\sin ky \sin lz$$

In addition, $V = 0$ when $y = \pi$ implies: $k = 1, 2, 3, \cdots$

and $V = 0$ when $z = \pi$ implies: $l = 1, 2, 3, \cdots$

$$V(x,y,z) = Ce^{-\sqrt{k^2+l^2}\,x} \sin ky \sin lz$$

$$V(x \to \infty) = 0$$



All the B.C.s are satisfied except the one at the left end.
We look for linear combination

$$\boxed{V(x,y,z) = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} C_{k,l} e^{-\sqrt{k^2+l^2}\,x} \sin ky \sin lz}$$

such that

$$V(0,y,z) = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} C_{k,l} \sin ky \sin lz = V_0(y,z)$$

$$V(0,y,z)=\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}C_{k,l}\sin ky\sin lz=V_0(y,z)$$

To evaluate the coefficients, multiply both sides by $\sin ny \sin mz$, where *m*, *n* are positive integers, and integrate:

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}C_{k,l}\int_0^{\pi}\sin ky\sin ny\,dy\int_0^{\pi}\sin lz\sin mz\,dy=\int_0^{\pi}\int_0^{\pi}V_0(y,z)\sin ny\sin mz\,dy\,dz$$

Again, since $\int_0^{\pi}\sin ky\sin ny\,dy=\dfrac{\pi}{2}\delta_{kn}$

we have

$$C_{n,m}=\left(\frac{2}{\pi}\right)^2\int_0^{\pi}\int_0^{\pi}V_0(y,z)\sin ny\sin mz\,dy\,dz$$

$$C_{n,m}=\left(\frac{2}{\pi}\right)^2\int_0^\pi\int_0^\pi V_0\left(y,z\right)\sin ny\sin mz\,dy\,dz$$

For instance, if $V_0\left(y,z\right)=V_0$ = constant, we have

$$C_{n,m}=\left(\frac{2}{\pi}\right)^2\int_0^\pi\int_0^\pi V_0\sin ny\sin mz\,dy\,dz=\frac{4V_0}{\pi^2}\int_0^\pi\sin ny\,dy\int_0^\pi\sin mz\,dz$$

$$=\frac{4V_0}{\pi^2 nm}\int_0^\pi\sin ny\,d\left(ny\right)\int_0^\pi\sin mz\,d\left(mz\right)=\frac{4V_0}{\pi^2 nm}\left[\cos ny\right]_0^\pi\left[\cos mz\right]_0^\pi$$

$$=\begin{cases}0 & \text{if } n \text{ or } m \text{ is even}\\ \dfrac{16V_0}{\pi^2 nm} & \text{if } n \text{ and } m \text{ are odd}\end{cases}$$

Hence,

$$V\left(x,y,z\right)=\frac{16V_0}{\pi^2}\sum_{n=1,3,5,\cdots}^{\infty}\sum_{m=1,3,5,\cdots}^{\infty}\frac{e^{-\sqrt{n^2+m^2}\,x}\sin ny\sin mz}{nm}$$

Spherical Coordinates

Laplace's equation:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0$$

Here we shall only consider the particular case in which there is azimuthal symmetry so that *V* is independent of ϕ

Look for solutions of the form

$$V(r,\theta)=R(r)\Theta(\theta)$$

Sub. $V(r,\theta)=R(r)\Theta(\theta)$ into the Laplace's equation, we have

$$\Theta\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+R\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=0$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=0$$

$$\begin{cases}\dfrac{1}{R}\dfrac{d}{dr}\left(r^2\dfrac{dR}{dr}\right)=\lambda\\[2ex]\dfrac{1}{\Theta\sin\theta}\dfrac{d}{d\theta}\left(\sin\theta\dfrac{d\Theta}{d\theta}\right)=-\lambda\end{cases}$$

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=-\lambda$$

For the angular part, let $x=\cos\theta \rightarrow$

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=-\frac{1}{\Theta}\frac{1}{dx/d\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dx}{d\theta}\frac{d\Theta}{dx}\right)$$

$$=\frac{1}{\Theta}\frac{d}{dx}\left(\sin^2\theta\frac{d\Theta}{dx}\right)=\frac{1}{\Theta}\frac{d}{dx}\left(\left(1-x^2\right)\frac{d\Theta}{dx}\right)=-\lambda$$

$$\Rightarrow\left(1-x^2\right)\frac{d^2\Theta}{dx^2}-2x\frac{d\Theta}{dx}+\lambda\Theta=0$$

The above differential equation admits solutions which are finite for all θ, only when

$$\lambda = l(l+1)$$

where l are non-negative integers. The solutions are called the Legendre polynomials

$$\Theta(\theta) = P_l(\cos\theta)$$

which can be obtained by the Rodrigues formula

$$P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2 - 1)^l$$

Here are the first 6 Rodrigues formulae

$$P_0(x) = 1$$

$$P_1(x) = x$$

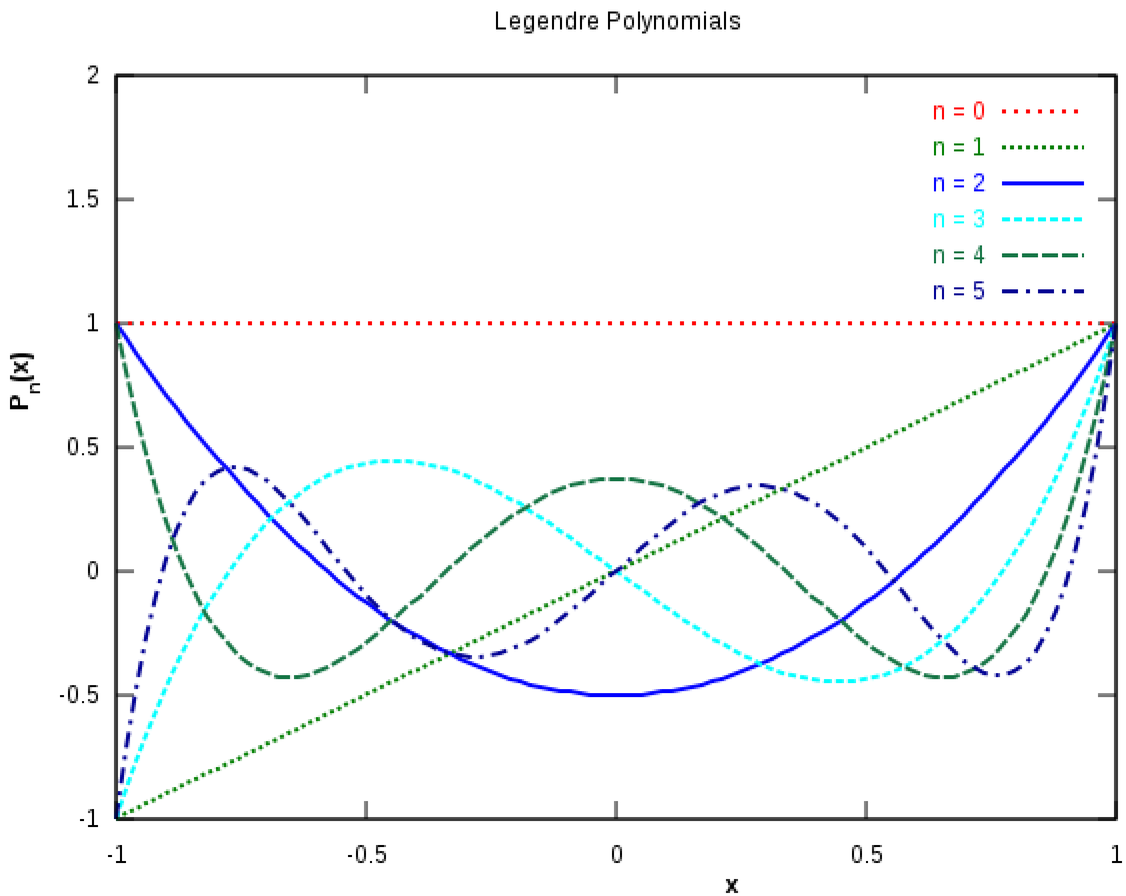
$$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$$

$$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$$

$$P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$$

$$P_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right)$$

$$\vdots$$



http://www.jimrolf.com/java.htm

The Legendre polynomials are complete and orthogonal in the interval $-1 \leq x \leq 1$.

The orthogonality can be proven by noticing that

$$\int_{-1}^{1} P_l(x) P_m(x) dx = \frac{2}{2m+1} \delta_{lm}$$

If $x = \cos\theta$, the polynomials are orthogonal in the interval $0 \leq \theta \leq \pi$:

$$\int_{0}^{\pi} P_l(\cos\theta) P_m(\cos\theta) \sin\theta d\theta = \frac{2}{2m+1} \delta_{lm}$$

$$
\begin{cases}
\dfrac{1}{R}\dfrac{d}{dr}\left(r^2\dfrac{dR}{dr}\right)=l(l+1) \\[2ex]
\dfrac{1}{\Theta\sin\theta}\dfrac{d}{d\theta}\left(\sin\theta\dfrac{d\Theta}{d\theta}\right)=-l(l+1)
\end{cases}
$$

The general solution of the radial equation is

$$
R(r)=Ar^l+\frac{B}{r^{l+1}}
$$

In conclusion, we have

$$V(r,\theta) = \left(Ar^l + \frac{B}{r^{l+1}} \right) P_l(\cos\theta)$$

and the general solution is

$$\boxed{V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)}$$

Example:

The potential $V_0(\theta)$ is specified on the surface of a hollow, empty sphere of radius *R*. Find the potential inside and outside the sphere if given that the potential at infinity is zero.

Inside:

The general solution is $$\boxed{V(r,\theta)=\sum_{l=0}^{\infty}\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)}$$

V being finite at the $r = 0$ implies: $B_l = 0$

So
$$V(r,\theta)=\sum_{l=0}^{\infty}A_l r^l P_l(\cos\theta)$$

At $r = R$,
$$V(R,\theta)=\sum_{l=0}^{\infty}A_l R^l P_l(\cos\theta)=V_0(\theta)$$

$$V(R,\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = V_0(\theta)$$

To evaluate the coefficients, multiply both sides by $P_m(\cos\theta)\sin\theta$ and integrate from 0 to π :

$$\sum_{l=0}^{\infty} \int_0^{\pi} A_l R^l P_l(\cos\theta) P_m(\cos\theta)\sin\theta d\theta = \int_0^{\pi} V_0(\theta) P_m(\cos\theta)\sin\theta d\theta$$

$$\sum_{l=0}^{\infty} A_l R^l \frac{2}{2m+1}\delta_{lm} = \int_0^{\pi} V_0(\theta) P_m(\cos\theta)\sin\theta d\theta$$

$$A_m R^m \frac{2}{2m+1} = \int_0^{\pi} V_0(\theta) P_m(\cos\theta)\sin\theta d\theta$$

$$A_m = \frac{2m+1}{2R^m}\int_0^{\pi} V_0(\theta) P_m(\cos\theta)\sin\theta d\theta$$

In particular, if $V_0(\theta) = \text{Constant} = V_0$, then

$$A_m = \frac{2m+1}{2R^m}\int_0^\pi V_0 P_m(\cos\theta)\sin\theta d\theta$$

$$= \frac{2m+1}{2R^m}V_0\int_0^\pi 1\cdot P_m(\cos\theta)\sin\theta d\theta$$

$$= \frac{2m+1}{2R^m}V_0\int_0^\pi P_0(\cos\theta)P_m(\cos\theta)\sin\theta d\theta$$

$$= \frac{2m+1}{2R^m}V_0\cdot 2\delta_{m0} = V_0\delta_{m0}$$

Hence,

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = V_0 r^0 P_0(\cos\theta) = V_0$$

Outside:

The general solution is $\boxed{V(r,\theta)=\sum_{l=0}^{\infty}\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)}$

$V\to 0$ when $r\to\infty$ implies: $A_l=0$

So

$$V(r,\theta)=\sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta)$$

At $r=R$,

$$V(R,\theta)=\sum_{l=0}^{\infty}\frac{B_l}{R^{l+1}}P_l(\cos\theta)=V_0(\theta)$$

$$V\left(R,\theta\right)=\sum_{l=0}^{\infty}\frac{B_l}{R^{l+1}}P_l\left(\cos\theta\right)=V_0\left(\theta\right)$$

To evaluate the coefficients, multiply both sides by $P_m\left(\cos\theta\right)\sin\theta$ and integrate from 0 to π :

$$\sum_{l=0}^{\infty}\int_0^{\pi}\frac{B_l}{R^{l+1}}P_l\left(\cos\theta\right)P_m\left(\cos\theta\right)\sin\theta d\theta=\int_0^{\pi}V_0\left(\theta\right)P_m\left(\cos\theta\right)\sin\theta d\theta$$

$$\sum_{l=0}^{\infty}\frac{B_l}{R^{l+1}}\frac{2}{2m+1}\delta_{lm}=\int_0^{\pi}V_0\left(\theta\right)P_m\left(\cos\theta\right)\sin\theta d\theta$$

$$\frac{B_m}{R^{m+1}}\frac{2}{2m+1}=\int_0^{\pi}V_0\left(\theta\right)P_m\left(\cos\theta\right)\sin\theta d\theta$$

$$B_m=\frac{2m+1}{2}R^{m+1}\int_0^{\pi}V_0\left(\theta\right)P_m\left(\cos\theta\right)\sin\theta d\theta$$

In particular, if $V_0\left(\theta\right)=\mathrm{Constant}=V_0$, then

$$B_m=\frac{2m+1}{2}R^{m+1}\int_0^{\pi}V_0P_m\left(\cos\theta\right)\sin\theta d\theta$$

$$=\frac{2m+1}{2}R^{m+1}V_0\int_0^{\pi}P_0\left(\cos\theta\right)P_m\left(\cos\theta\right)\sin\theta d\theta$$

$$=\frac{2m+1}{2}R^{m+1}V_0\cdot 2\delta_{m0}=RV_0\delta_{m0}$$

Hence,

$$V\left(r,\theta\right)=\sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l\left(\cos\theta\right)=\frac{B_0}{r}P_0\left(\cos\theta\right)=V_0\frac{R}{r}$$

Example:

An uncharged metal sphere of radius R is placed in an otherwise uniform electric field $\mathbf{E} = E_0\hat{\mathbf{k}}$. The field will push positive charge to the "northern" surface of the sphere, and negative charge to the "southern" surface. This induced charge, in turn, distorts the field in the neighborhood of the sphere.

What is the potential in the region outside the sphere?

Solution:

The surface of the metal sphere is an equipotential. This potential can be arbitrarily set to zero. Therefore, at large distance, the potential varies as $V \to -E_0 z$

Hence, the B.C.s are

(i) $V\left(R,\theta\right)=0$

(ii) $V \to -E_0 r\cos\theta$ when $r >> R$

The general form of the potential is

$$V(r,\theta)=\sum_{l=0}^{\infty}\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

Hence, (i) implies

$$A_l R^l+\frac{B_l}{R^{l+1}}=0$$

$$\Rightarrow B_l=-A_l R^{2l+1}$$

That is

$$V(r,\theta)=\sum_{l=0}^{\infty}A_l\left(r^l-\frac{R^{2l+1}}{r^{l+1}}\right)P_l(\cos\theta)$$

When $r >> R$, the second term in the bracket is negligible and (ii) implies

$$\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = -E_0 r\cos\theta = -E_0 r P_1(\cos\theta)$$

Therefore,

$$A_1 = -E_0$$

and all other coefficients are zero.

The potential is hence

$$V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta$$

The induced surface charge can also be found by

$$\sigma(\theta) = -\varepsilon_0 \left.\frac{\partial V}{\partial r}\right|_{r=R}$$

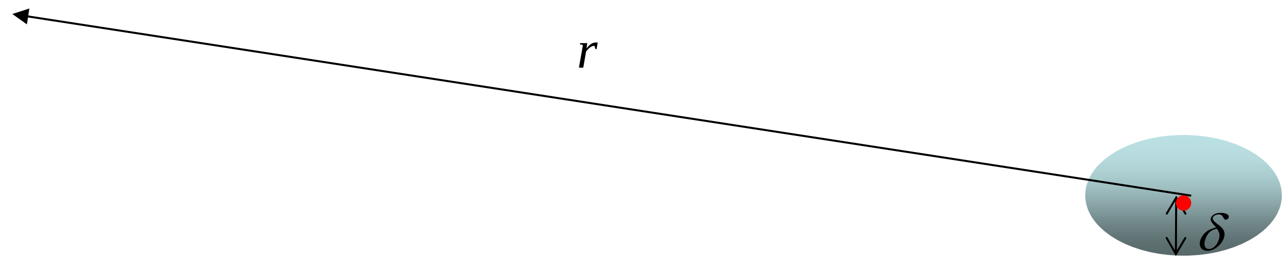
$$= \varepsilon_0 E_0 \left(1 + 2\frac{R^3}{r^3}\right)\cos\theta \bigg|_{r=R}$$

$$= 3\varepsilon_0 E_0 \cos\theta$$

Special Techniques for Calculating Potentials

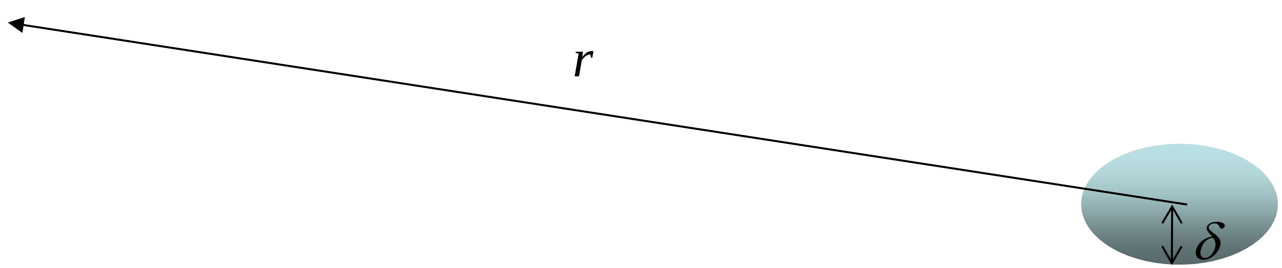
Multipole Expansion

- It is clear that a localized charge distribution $\rho(\mathbf{r})$ carrying net charge Q, when observed at a distance r very large compared with its size δ, is approximately the same as a point charge Q
- The finite size of the distribution leads to correction terms to the Coulomb potential or field of a point charge



- The multipole expansion is the series expansion of the potentials (and hence the fields) in ascending order of δ/r, with which one can obtain approximations of the potentials and fields to various orders
- The approximations is good when

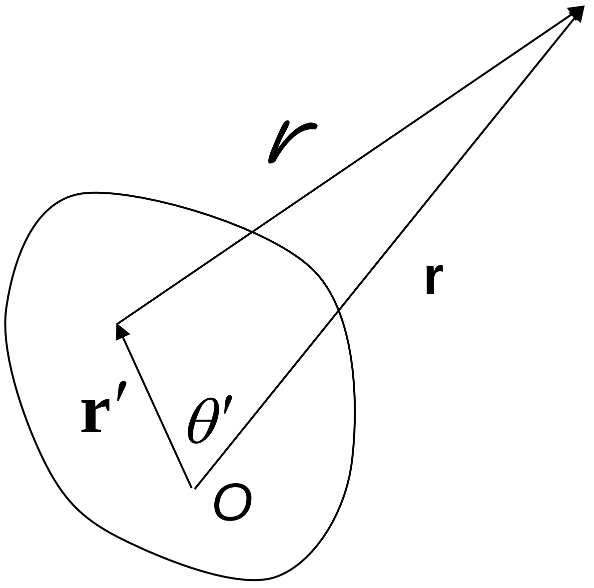
$$r \gg \delta$$



Multipole Expansion of the Electric Potential

- Consider a localized charge distribution given by $\rho(\mathbf{r}')$
- The potential at **r** is

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{\mathscr{r}}\rho(\mathbf{r}')\,d\tau'$$



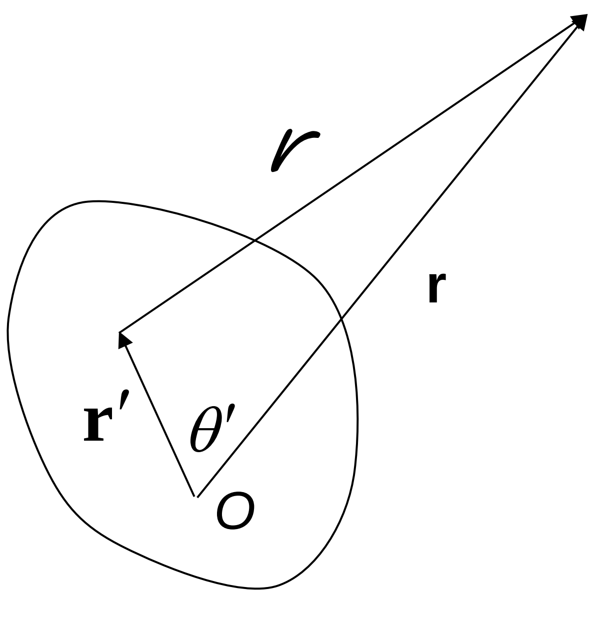
- From the law of cosines

$$\mathscr{r}^2 = r^2 + r'^2 - 2rr'\cos\theta'$$

- Hence

$$\mathscr{r} = r\sqrt{1+\left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\cos\theta'} = r\sqrt{1+\varepsilon}$$

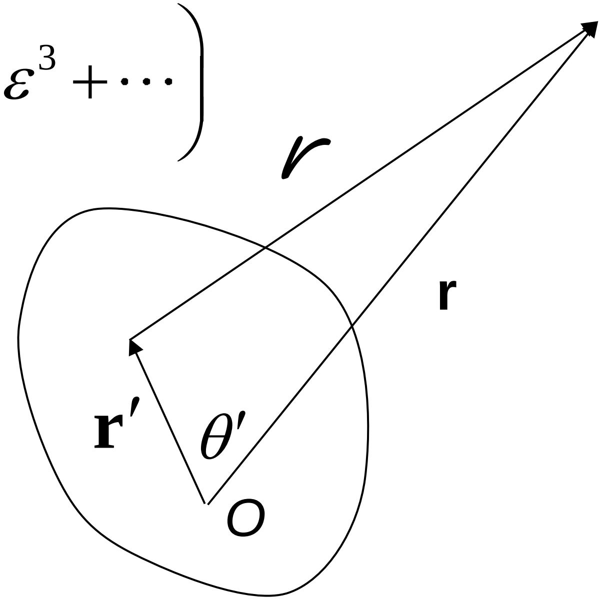
where $\varepsilon = \dfrac{r'}{r}\left(\dfrac{r'}{r} - 2\cos\theta'\right)$



- Since the charge is localized, if we are interested in the potential at a point **r** far away from the charges, such that for all **r'** with non-zero ρ, $r \gg r'$

then
$$\varepsilon = \frac{r'}{r}\left(\frac{r'}{r} - 2\cos\theta'\right) \ll 1 \longrightarrow \mathscr{r} = r\sqrt{1+\varepsilon}$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}(1+\varepsilon)^{-1/2} = \frac{1}{r}\left(1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 - \frac{5}{16}\varepsilon^3 + \cdots\right)$$



$$\frac{1}{\mathscr{r}}=\frac{1}{r}\left(1-\frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r}-2\cos\theta'\right)+\frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r}-2\cos\theta'\right)^2-\frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r}-2\cos\theta'\right)^3+\cdots\right)$$

$$=\frac{1}{r}\left(1+\left(\frac{r'}{r}\right)\cos\theta'+\left(\frac{r'}{r}\right)^2\left(3\cos^2\theta'-1\right)/2+\left(\frac{r'}{r}\right)^3\left(5\cos^3\theta'-3\cos\theta'\right)/2+\cdots\right)$$

- It can be proved that if one collects the terms according to the powers in *r*/*r'*, then the coefficient of the term (*r*/*r'*)^n is just exactly the Legendre polynomials, i.e.

$$\frac{1}{\mathscr{r}}=\frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n\left(\cos\theta'\right)$$

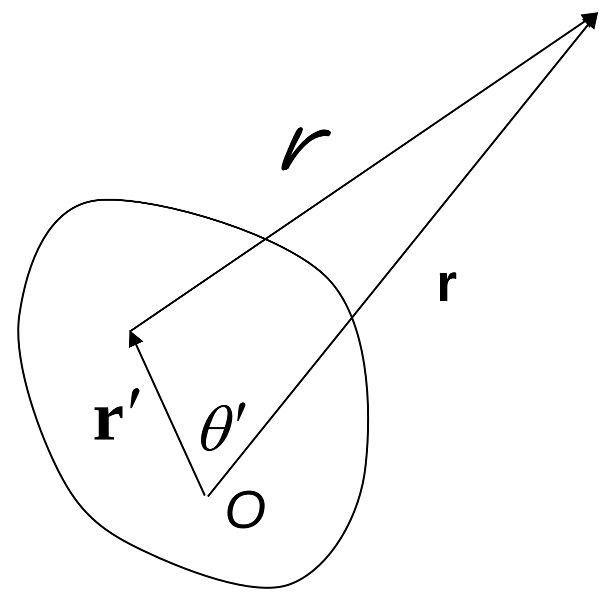
$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^{n} P_n\left(\cos\theta'\right)$$

- The potential at **r** is therefore

$$V\left(\mathbf{r}\right) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{\mathscr{r}}\rho\left(\mathbf{r}'\right)d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int r'^{n} P_n\left(\cos\theta'\right)\rho\left(\mathbf{r}'\right)d\tau'$$

- The above equation is exact
- Keeping a finite number of terms up to certain order gives the approximate potential at large distance



Monopole Term $n = 0$

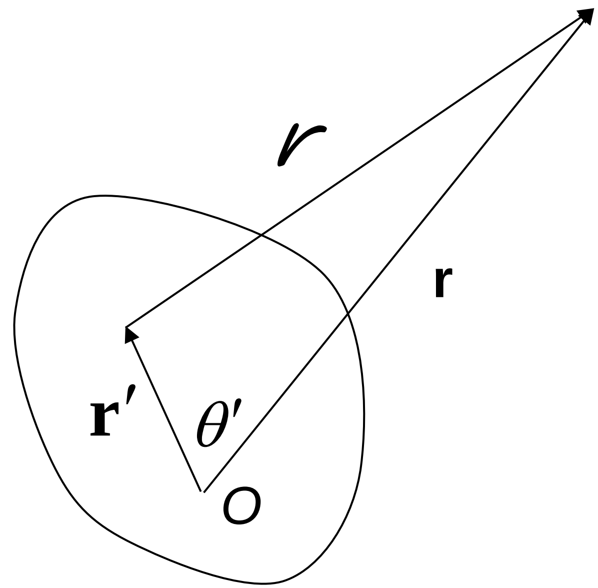
$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{\mathscr{r}}\rho(\mathbf{r}')\,d\tau' = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int r'^{n}P_n(\cos\theta')\rho(\mathbf{r}')\,d\tau'$$

$$\downarrow$$

$$\frac{1}{\mathscr{r}} \to \frac{1}{r}$$

$$\downarrow$$

$$V_{\text{mon}}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\int \rho(\mathbf{r}')\,d\tau'$$



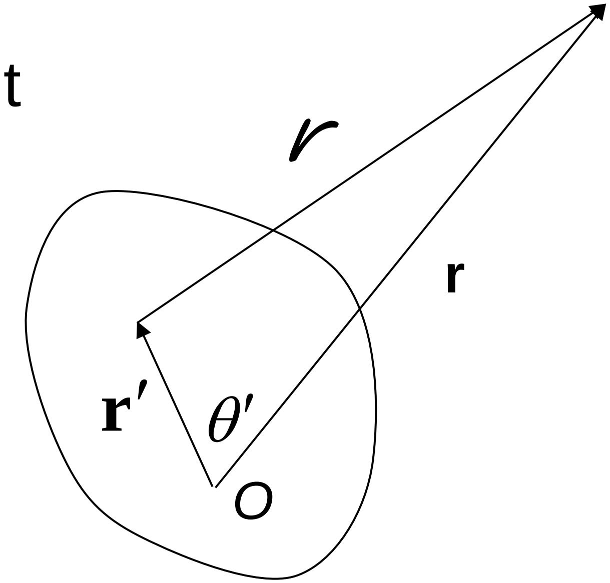
Monopole Term $n = 0$

$$V_{\text{mon}}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\int \rho(\mathbf{r}')\,d\tau'$$

$Q = \int \rho(\mathbf{r}')\,d\tau'$ is the total charge of the distribution

It is also called the monopole moment

Notice that the monopole moment Q is obviously independent of the point we choose as the origin



- This is the dominant term at large distance if $Q \neq 0$
- Under this approximation, all the charges are considered to be located at the origin
- The monopole term of the potential goes like $1/r$:

$$V_{\mathrm{mon}}(\mathbf{r}) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r}$$

- The corresponding **E** field goes like $1/r^2$:

$$\mathbf{E}_{\mathrm{mon}} = -\nabla V_{\mathrm{mon}}(\mathbf{r}) = -\nabla\left(\frac{Q}{4\pi\varepsilon_0}\frac{1}{r}\right) = \frac{Q}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}}{r^2}$$

Dipole Term *n* = 1

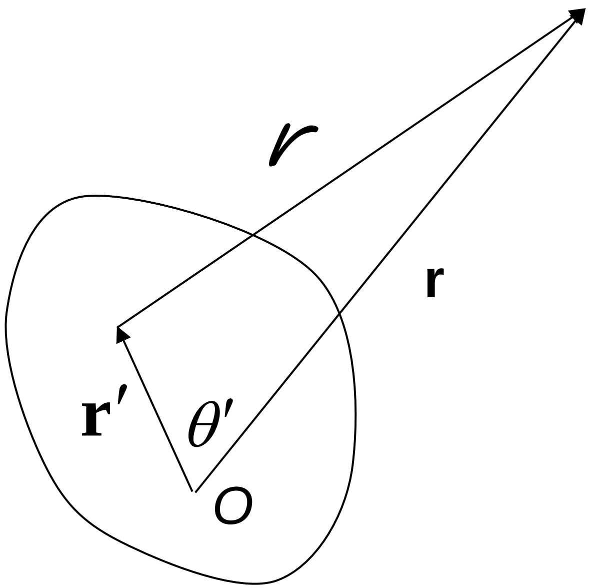
- When $Q = 0$, the dominant term at large r is the dipole term:

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{r}\rho(\mathbf{r}')\,d\tau' = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int r'^n P_n(\cos\theta')\,\rho(\mathbf{r}')\,d\tau'$$

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\int r'\cos\theta'\rho(\mathbf{r}')\,d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\int \hat{\mathbf{r}}\cdot\mathbf{r}'\rho(\mathbf{r}')\,d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}}{r^2}\cdot\int \mathbf{r}'\rho(\mathbf{r}')\,d\tau'$$



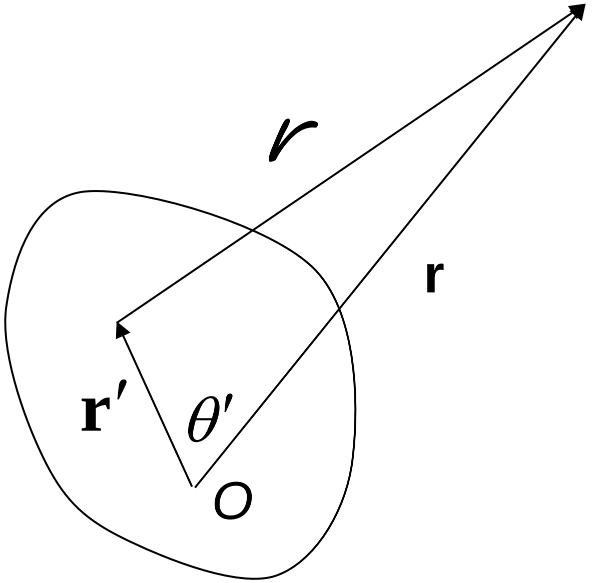
$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}}{r^2}\cdot\int \mathbf{r}'\rho(\mathbf{r}')d\tau'$$

- Let us define the <u>**dipole moment**</u> of the distribution by

$$\mathbf{p} = \int \mathbf{r}'\rho(\mathbf{r}')d\tau'$$

- Then

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$



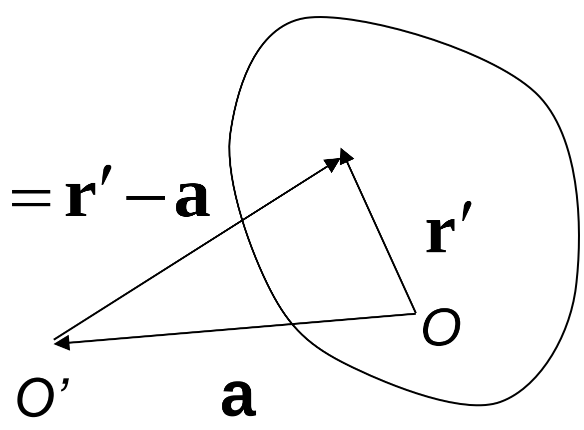
$$\mathbf{p} = \int \mathbf{r}' \rho(\mathbf{r}') d\tau'$$

- The dipole moment so defined is a vector. It depends on *the choice of origin*
- If we displace the origin by **a**

  then the new dipole moment is

$$\begin{aligned}
\bar{\mathbf{p}} &= \int \bar{\mathbf{r}}' \rho(\bar{\mathbf{r}}') d\bar{\tau}' \\
&= \int (\mathbf{r}' - \mathbf{a}) \rho(\mathbf{r}') d\tau' \\
&= \int \mathbf{r}' \rho(\mathbf{r}') d\tau' - \mathbf{a} \int \rho(\mathbf{r}') d\tau' \\
&= \mathbf{p} - Q\mathbf{a}
\end{aligned}$$

$$\bar{\mathbf{r}}' = \mathbf{r}' - \mathbf{a}$$

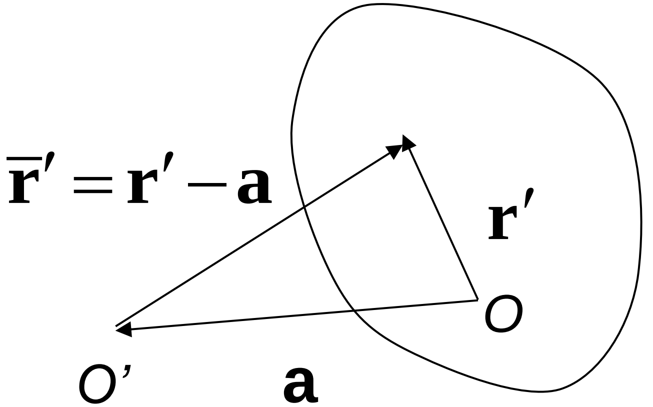


$$\bar{\mathbf{p}} = \mathbf{p} - Q\mathbf{a}$$

- However, if $Q = 0$, then the dipole moment becomes independent of the choice of origin:

$$\bar{\mathbf{p}} = \mathbf{p}$$

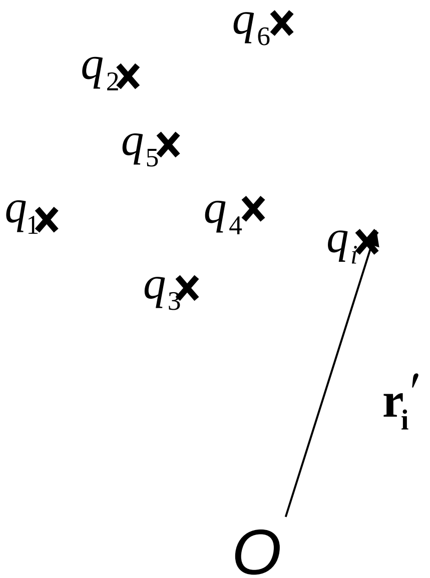
- It depends only on size, shape and distribution of the sources



$$\mathbf{p} = \int \mathbf{r}'\rho(\mathbf{r}')d\tau'$$

For *n* point charges $q_1, q_2, \cdots, q_n$ at $\mathbf{r}_1', \mathbf{r}_2', \cdots, \mathbf{r}_n'$, respectively,

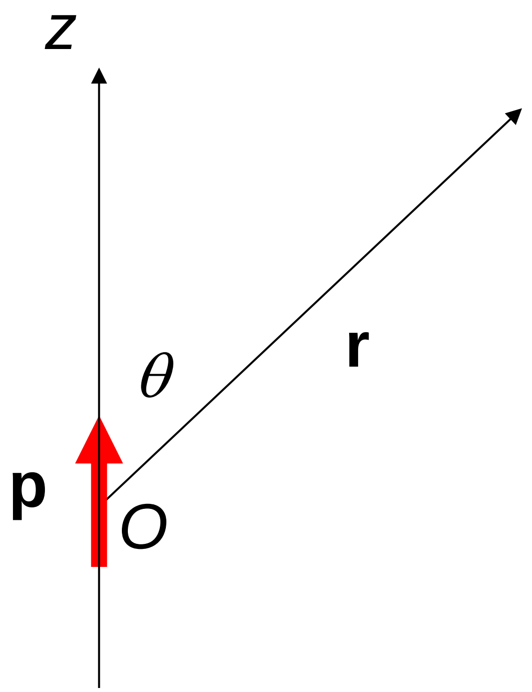
$$\mathbf{p} = \sum_{i=1}^{n} q_i \mathbf{r}_i'$$



- The dipole potential goes like $1/r^2$:

$$V_{\text{dip}}(\mathbf{r}) = \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2}$$

- If **p** is along the z direction: $V_{\text{dip}}(\mathbf{r}) = \dfrac{p\cos\theta}{4\pi\varepsilon_0 r^2}$



$$V_{\text{dip}}(\mathbf{r}) = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$

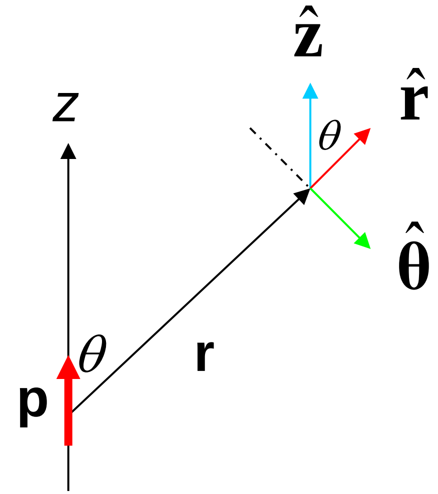
- The corresponding **E** field goes like 1/*r*^{3}:

$$\mathbf{E}_{\text{dip}} = -\nabla V_{\text{dip}}(\mathbf{r}) = -\hat{\mathbf{r}}\frac{\partial V_{\text{dip}}}{\partial r} - \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial V_{\text{dip}}}{\partial \theta} - \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial V_{\text{dip}}}{\partial \phi}$$

$$\mathbf{E}(r,\theta) = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right)$$

$$\mathbf{E}(r,\theta) = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right)$$

- The above expression makes reference to a particular coordinate system and direction of **p**
- To obtain a coordinate-free form:



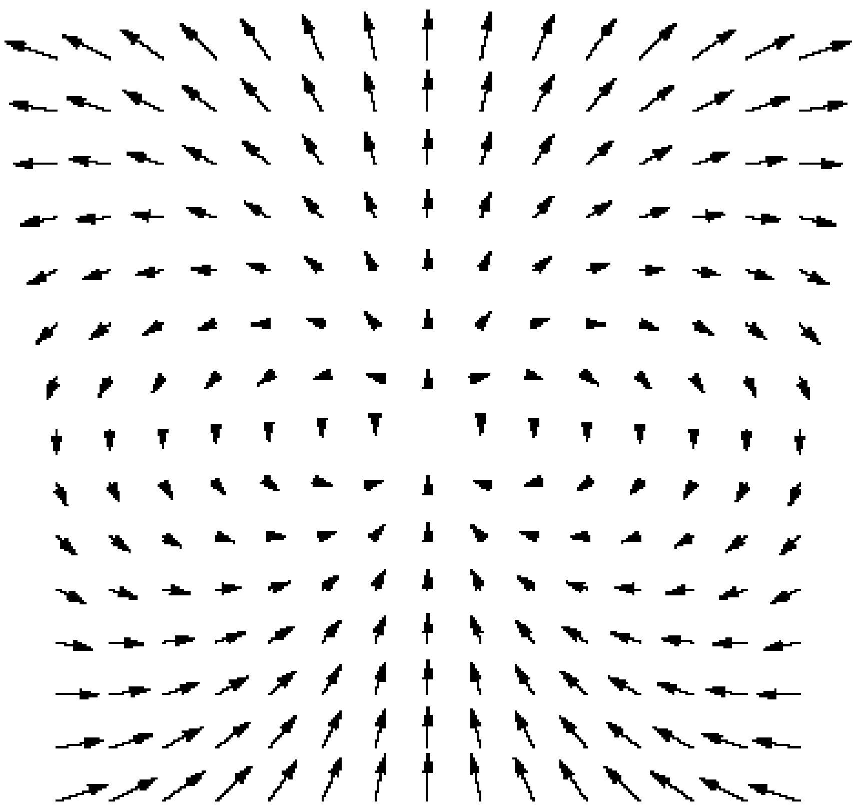
$$\mathbf{E} = \frac{p}{4\pi\varepsilon_0 r^3}\left(3\cos\theta\,\hat{\mathbf{r}} - \cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right)$$

$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

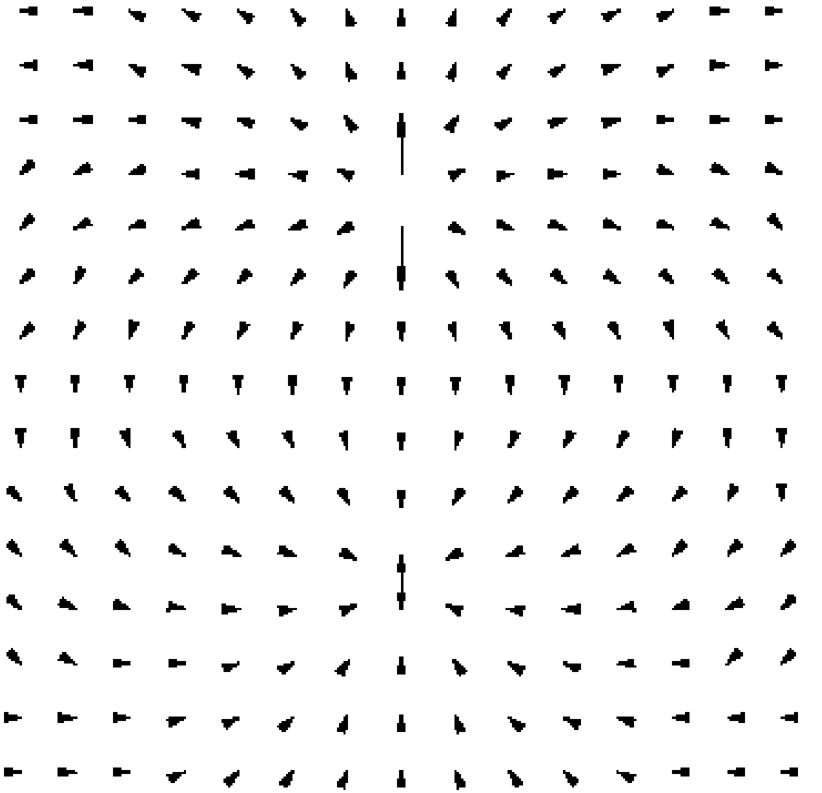
$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0 r^3}\left(3p\cos\theta\,\hat{\mathbf{r}} - p\hat{\mathbf{z}}\right)$$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\left(3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}\right)$$

E field of a pure dipole at the origin:



E field of a physical dipole:



Quadrupole Term $n = 2$

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{\mathscr{r}}\rho(\mathbf{r}')\,d\tau' = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int r'^n P_n(\cos\theta')\,\rho(\mathbf{r}')\,d\tau'$$

$$V_{\text{quad}}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\int r'^2\,\frac{1}{2}\left(3\cos^2\theta' - 1\right)\rho(\mathbf{r}')\,d\tau'$$

- The dominant term when both Q and $\mathbf{p}$ vanish, and the <u>quadrupole moment</u> is non-zero
- The potential goes like $1/r^3$
- The corresponding E field goes like $1/r^4$

Octopole Term $n = 3$

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{\kappa}\rho(\mathbf{r}')\,d\tau' = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int r'^n P_n(\cos\theta')\,\rho(\mathbf{r}')\,d\tau'$$

$$V_{\mathrm{octo}}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^4}\int r'^3\,\frac{1}{2}\left(5\cos^3\theta' - 3\cos\theta'\right)\rho(\mathbf{r}')\,d\tau'$$

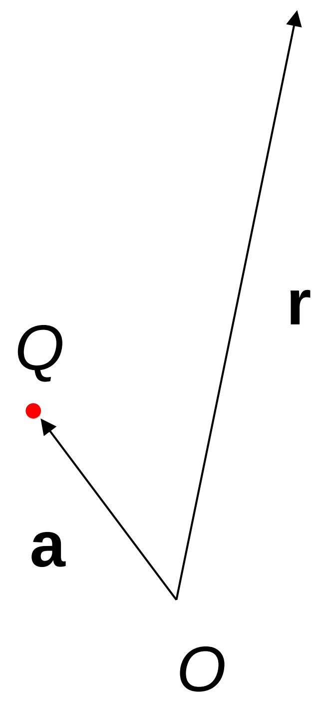
- The dominant term when both Q, $\mathbf{p}$, and the quadrupole moment vanish, while the <u>octopole moment</u> is non-zero
- The potential goes like $1/r^4$
- The corresponding E field goes like $1/r^5$

Pure Monopole

- A pure monopole is a distribution which gives only the monopole term
- Consider a point charge *Q* at **a**
- The multipole expansion gives

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int r'^n P_n(\cos\theta')Q\delta(\mathbf{r}'-\mathbf{a})d\tau'$$

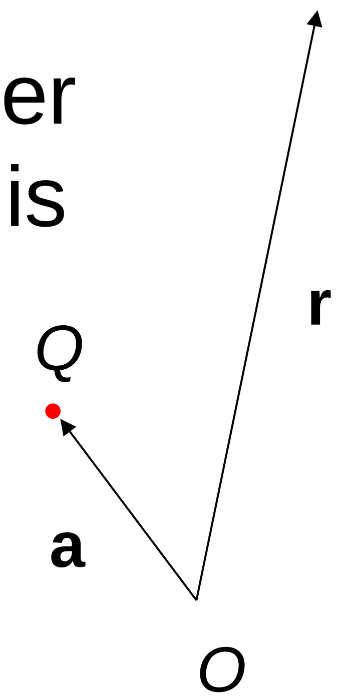
$$= \frac{Q}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{a^n P_n(\cos\theta')}{r^{n+1}}$$



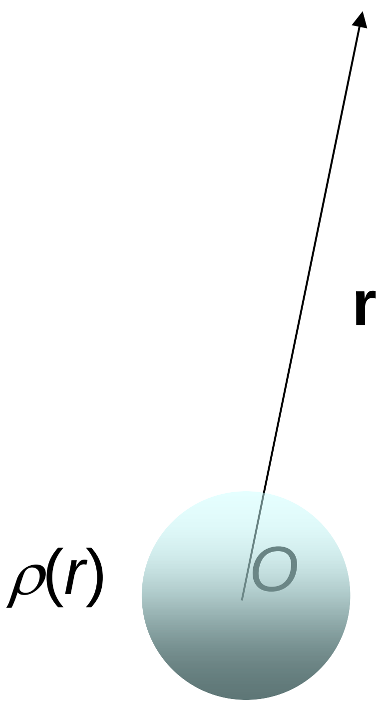
$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int r'^n P_n(\cos\theta')Q\delta(\mathbf{r}'-\mathbf{a})d\tau'$$

$$= \frac{Q}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{a^n P_n(\cos\theta')}{r^{n+1}}$$

- Notice that although we have a single point charge, there are higher order moments unless the charge is at the origin, so that $a = 0$
- Hence a point charge at the origin gives a pure monopole



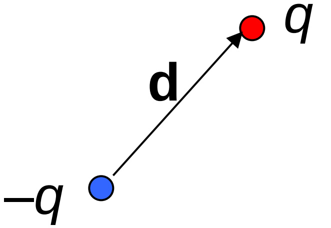
- Another example is a localized, spherically symmetric charge distribution $\rho(r)$
- The field at a far distance is exactly the same as that of a point charge at the origin.



Pure Dipole

- A pure dipole is an idealized distribution which gives <u>only</u> the dipole term
- To give a zero monopole term, $Q = 0$
- Therefore, the dipole moment of a pure dipole is independent of the origin

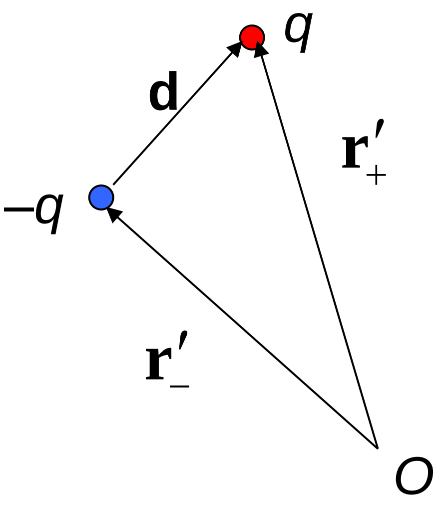
- Consider a pair of point charges q and $-q$ separated by distance d
- The monopole moment (total charge) is zero
- Hence, at far distance, the dominant term is the dipole term



- Consider two point charges q and $-q$, located at $\mathbf{r'}_+$ and $\mathbf{r'}_-$, respectively
- The dipole moment is

$$\mathbf{p} = q\mathbf{r}'_+ - q\mathbf{r}'_- = q\left(\mathbf{r}'_+ - \mathbf{r}'_-\right) = q\mathbf{d}$$

$$\mathbf{d} = \mathbf{r}'_+ - \mathbf{r}'_-$$



- For a physical dipole with non-zero d, higher order moments are non-zero

- To obtain a pure dipole, let d → 0 and q → ∞, while keeping $p = qd$ finite
- Because higher order moments have the orders qd^2, qd^3, …, all of them will vanish under this limit
- This is the simplest way to depict a pure dipole