Course Webpage:

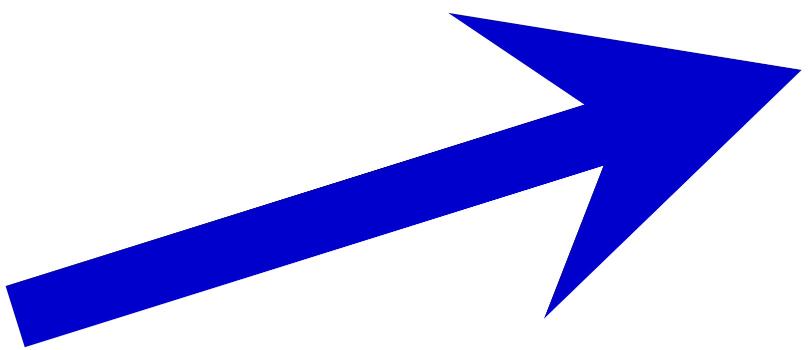
http://teaching.phys.ust.hk/Phys3033/

http://teaching.phys.ust.hk/Phys3053/

Vector Analysis

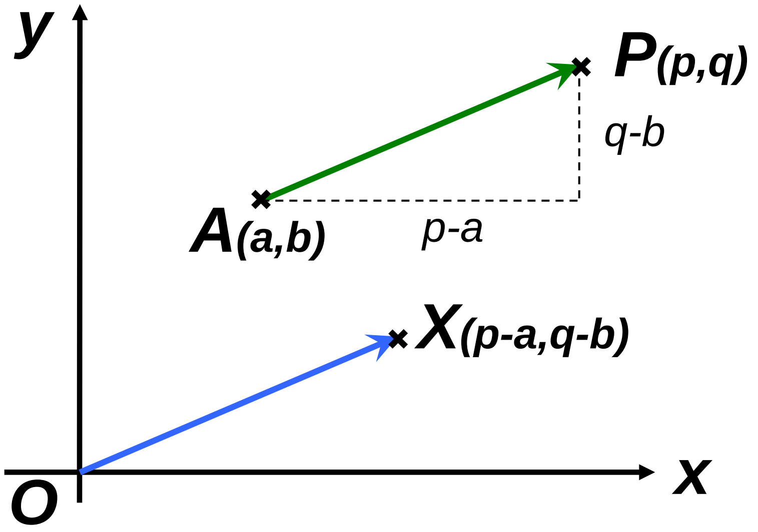
What is a vector?

A vector in n dimension is any set of n-components that transforms in the same manner as a displacement when you change coordinates



Displacement **is the model for the behavior of all vectors**

e.g. In 2D,



$$\overrightarrow{AP} = \overrightarrow{OX}$$

vector **:** ***direction*** **+** ***magnitude***

What is scalar?

A scalar has no direction and remains unchanged when one changes the coordinates.

Example:
Scalars: mass, charge, density, temperature
Vectors: velocity, acceleration, force, momentum

Notation:

vector – **Bold face** $\mathbf{A}$, in handwriting $\vec{A}$
scalar – ordinary character A

The magnitude of the vector is denoted by:

$$A = |\mathbf{A}| = \left|\vec{A}\right|$$

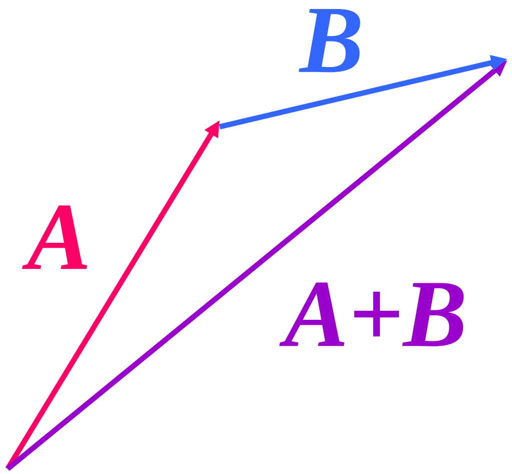
Vector Algebra

Vector Operations

(a) Addition of two vectors

Parallelogram law:

To find $\mathbf{A}+\mathbf{B}$*, place the tail of* B *at the head of* $\mathbf{A}$ *and draw the vector from the tail of* $\mathbf{A}$ *to the head of* $\mathbf{B}$



From the definition, the addition of vectors is

(i)Commutative

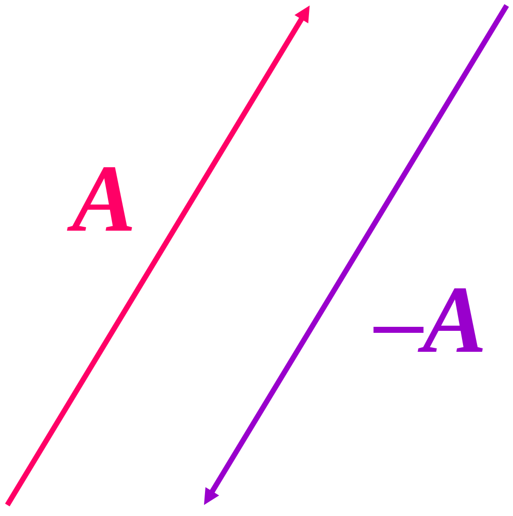
$$\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$$

(ii)Associative

$$(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$$

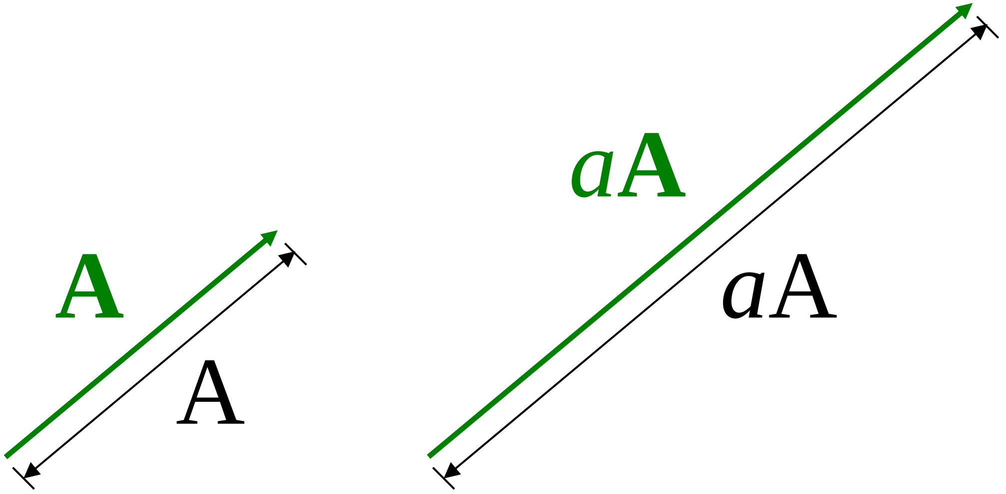
(b) Negative of a vector

The negative of a vector is defined as the vector with the same magnitude but opposite direction:



(c) Multiplication by a scalar

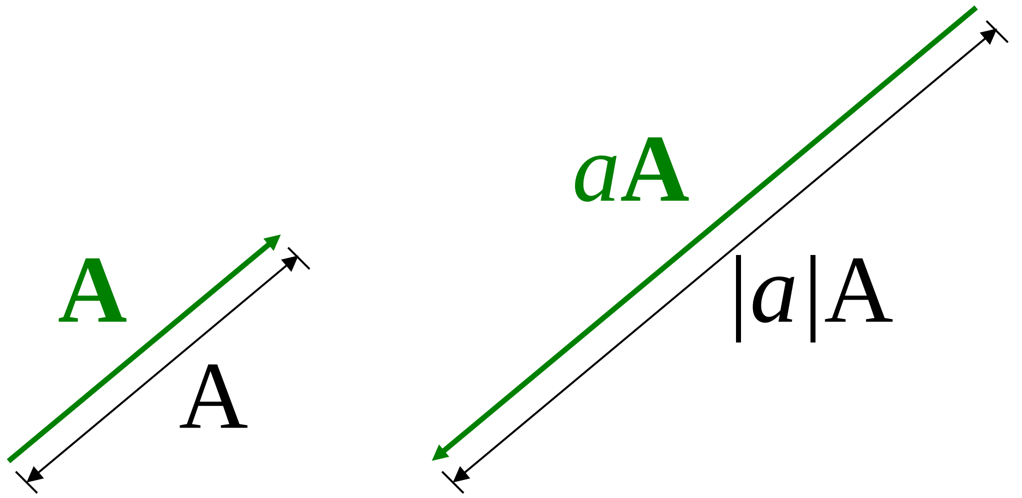
Multiplication by a positive real number a multiplies the magnitude by a times while leaving the direction unchanged.



(c) Multiplication by a scalar

Multiplication by a negative real number a **is defined by**

$$a\mathbf{A} = |a|(-\mathbf{A})$$



(c) Multiplication by a scalar

Multiplication by 0 gives the null vector 0 which satisfies

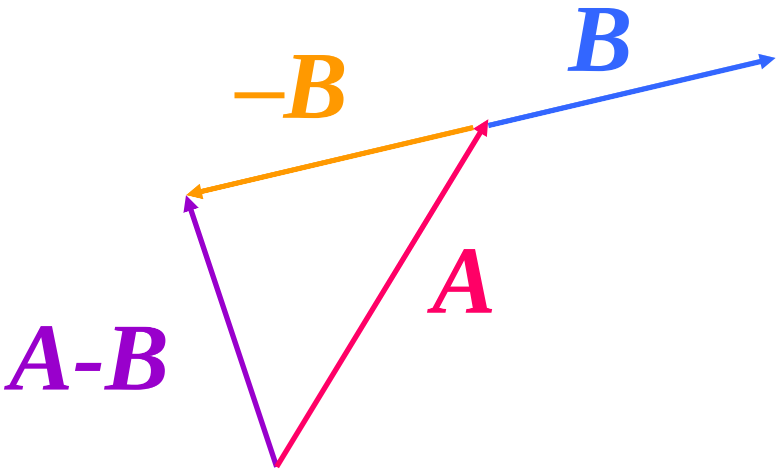
$$\boldsymbol{A}+\boldsymbol{0}=\boldsymbol{A}$$

Multiplication is distributive, i.e.,

$$a(\mathbf{A}+\mathbf{B})=a\mathbf{A}+a\mathbf{B}$$

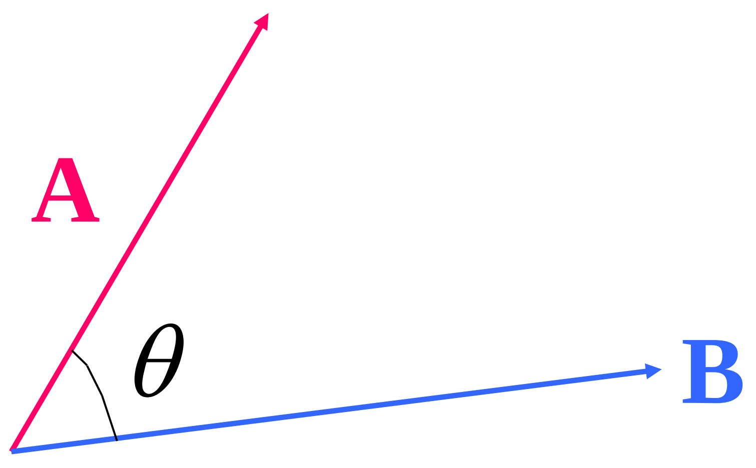
(d) Subtraction of two vectors

A-B **is defined by** ***A+(-B)***



(e) Dot product of two vectors

$$\vec{A}\cdot\vec{B} = AB\cos\theta$$



Dot product is also called scalar product or inner product

(e) Dot product of two vectors

Dot product is commutative:

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$$

Dot product is distributive:

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$$

(can u proof it?)

(e) Dot product of two vectors

If A, B perpendicular,

$$\theta = \pi / 2 \rightarrow \mathbf{A} \cdot \mathbf{B} = 0$$

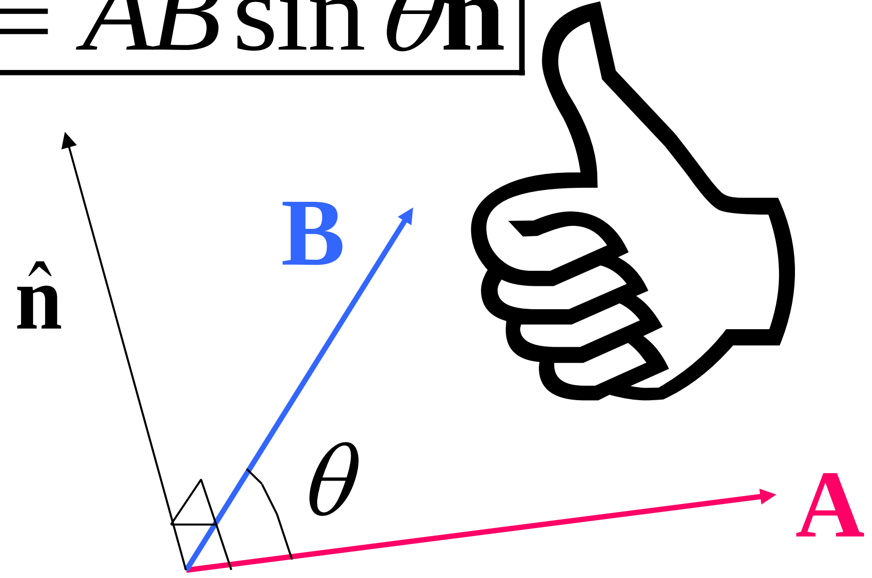
If A, B point to the same direction,

$$\theta = 0 \rightarrow \mathbf{A} \cdot \mathbf{B} = AB$$

In particular, $\mathbf{A} \cdot \mathbf{A} = A^2$

(f) Cross product of two vectors

$$\mathbf{A} \times \mathbf{B} = AB \sin\theta \, \hat{\mathbf{n}}$$



$\hat{\mathbf{n}}$ **is a unit vector (magnitude =1) perpendicular to the plane spanned by A and B, with direction (up or down) determined by the right-hand rule**

(f) Cross product of two vectors

Cross product is distributive:

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$$

(can u proof it?)

Cross product is <u>NOT</u> commutative:

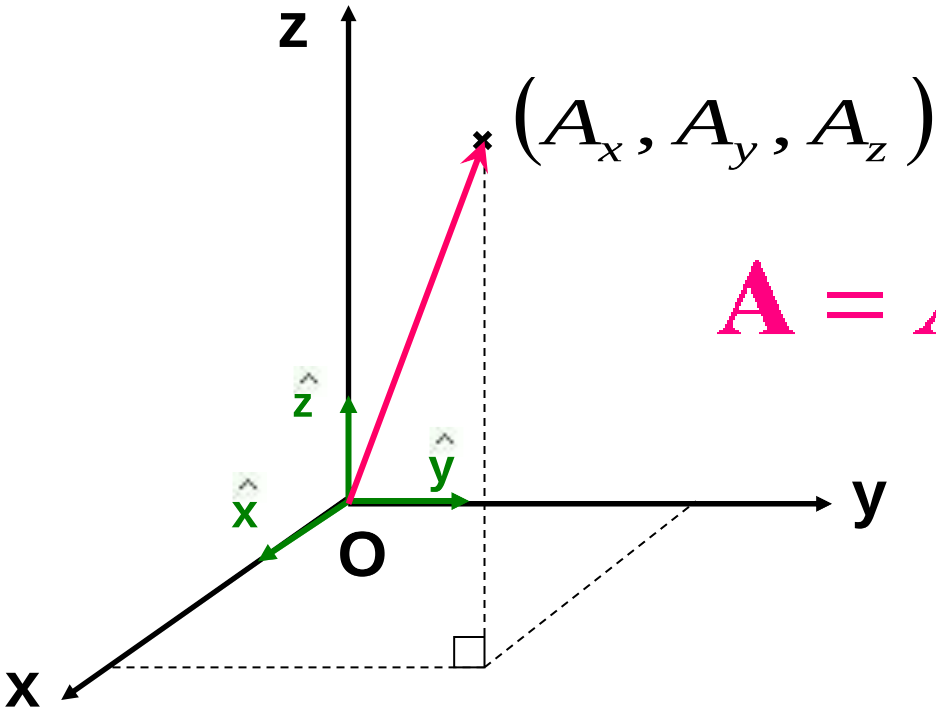
$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$

In particular, $\mathbf{A} \times \mathbf{A} = 0$

Vector Algebra

Component Form

Using Cartesian coordinate, with unit vectors $\hat{\mathrm{x}}, \hat{\mathrm{y}}, \hat{\mathbf{z}}$ parallel to the *x*, *y*, and *z* axes, respectively.



$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$$

(a) To add vectors, add like components

$$\mathbf{A}+\mathbf{B}=\left(A_x\hat{\mathbf{x}}+A_y\hat{\mathbf{y}}+A_z\hat{\mathbf{z}}\right)+\left(B_x\hat{\mathbf{x}}+B_y\hat{\mathbf{y}}+B_z\hat{\mathbf{z}}\right)$$

$$\boxed{\mathbf{A}+\mathbf{B}=\left(A_x+B_x\right)\hat{\mathbf{x}}+\left(A_y+B_y\right)\hat{\mathbf{y}}+\left(A_z+B_z\right)\hat{\mathbf{z}}}$$

(b) To multiply by a scalar, multiply each component

$$\boxed{aA=\left(aA_x\right)\hat{\mathbf{x}}+\left(aA_y\right)\hat{\mathbf{y}}+\left(aA_z\right)\hat{\mathbf{z}}}$$

(c) Since

$$\hat{x}\cdot\hat{x} = \hat{y}\cdot\hat{y} = \hat{z}\cdot\hat{z} = 1$$

$$\hat{x}\cdot\hat{y} = \hat{y}\cdot\hat{z} = \hat{z}\cdot\hat{x} = 0$$

$$\longrightarrow \vec{A}\cdot\vec{B} = (A_x\hat{x} + A_y\hat{y} + A_z\hat{z})\cdot(B_x\hat{x} + B_y\hat{y} + B_z\hat{z})$$

$$\boxed{\vec{A}\cdot\vec{B} = (A_xB_x + A_yB_y + A_zB_z)}$$

(d)

$$\hat{\mathbf{x}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{z}} = 0$$

$$\hat{\mathbf{x}} \times \hat{\mathbf{y}} = -\hat{\mathbf{y}} \times \hat{\mathbf{x}} = \hat{\mathbf{z}}$$

$$\hat{\mathbf{y}} \times \hat{\mathbf{z}} = -\hat{\mathbf{z}} \times \hat{\mathbf{y}} = \hat{\mathbf{x}}$$

$$\hat{\mathbf{z}} \times \hat{\mathbf{x}} = -\hat{\mathbf{x}} \times \hat{\mathbf{z}} = \hat{\mathbf{y}} \rightarrow$$

$$\mathbf{A} \times \mathbf{B} = \left(A_x \hat{\mathbf{x}} + A_y \hat{\mathbf{y}} + A_z \hat{\mathbf{z}}\right) \times \left(B_x \hat{\mathbf{x}} + B_y \hat{\mathbf{y}} + B_z \hat{\mathbf{z}}\right)$$

$$= \left(A_y B_z - A_z B_y\right)\hat{\mathbf{x}} + \left(A_z B_x - A_x B_z\right)\hat{\mathbf{y}} + \left(A_x B_y - A_y B_x\right)\hat{\mathbf{z}}$$

$$\boxed{\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}}$$

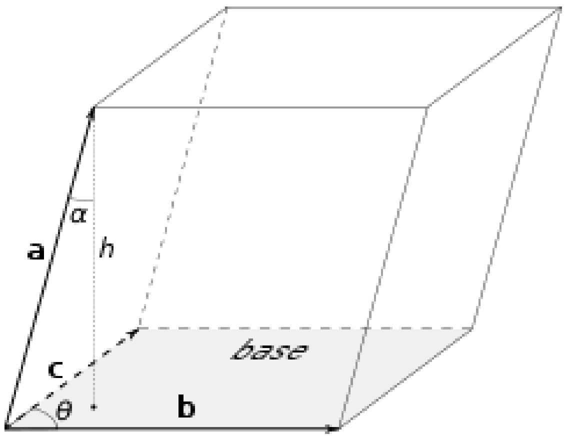
Triple Product

(a)

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \left(A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}\right)\cdot\left(\left(B_yC_z - B_zC_y\right)\hat{\mathbf{x}} + \left(B_zC_x - B_xC_z\right)\hat{\mathbf{y}} + \left(B_xC_y - B_yC_x\right)\hat{\mathbf{z}}\right)$$

$$= A_x\left(B_yC_z - B_zC_y\right) + A_y\left(B_zC_x - B_xC_z\right) + A_z\left(B_xC_y - B_yC_x\right)$$

$$\boxed{\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}}$$



(signed) volume of the parallelepiped defined by the three vectors

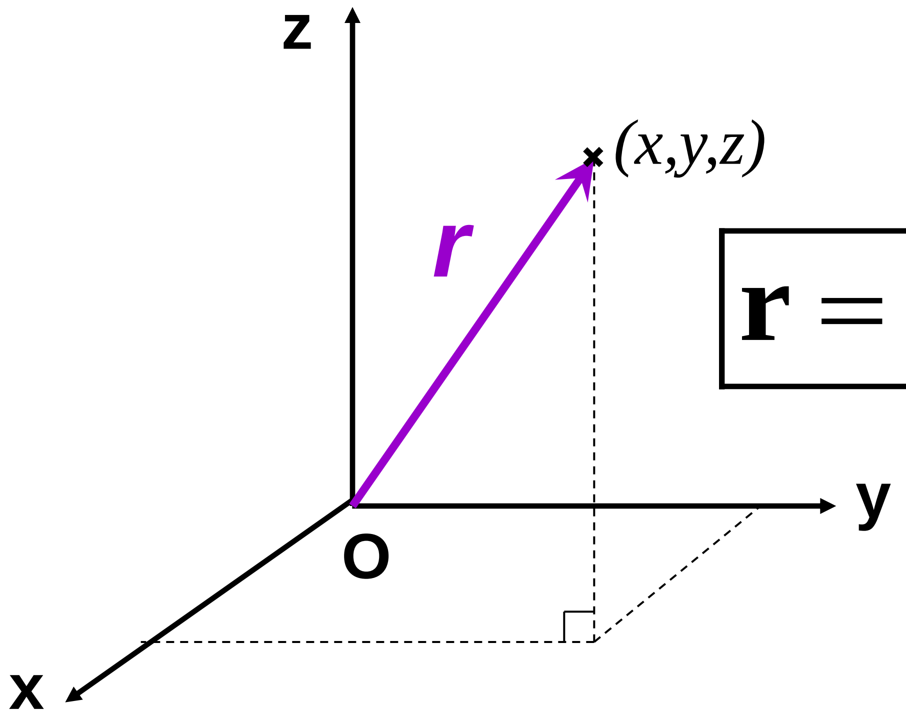
(b)

$$\boxed{\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})}$$

Vector Analysis

Position, Displacement, Separation Vectors

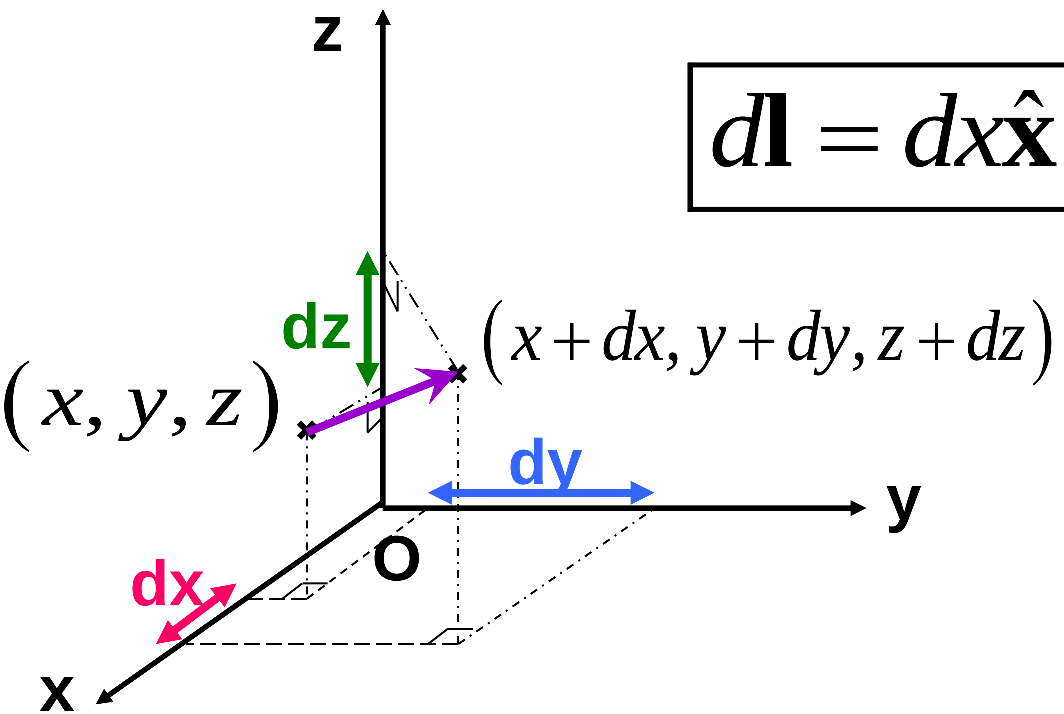
(a) Position vector pointing from origin to (x, y, z):



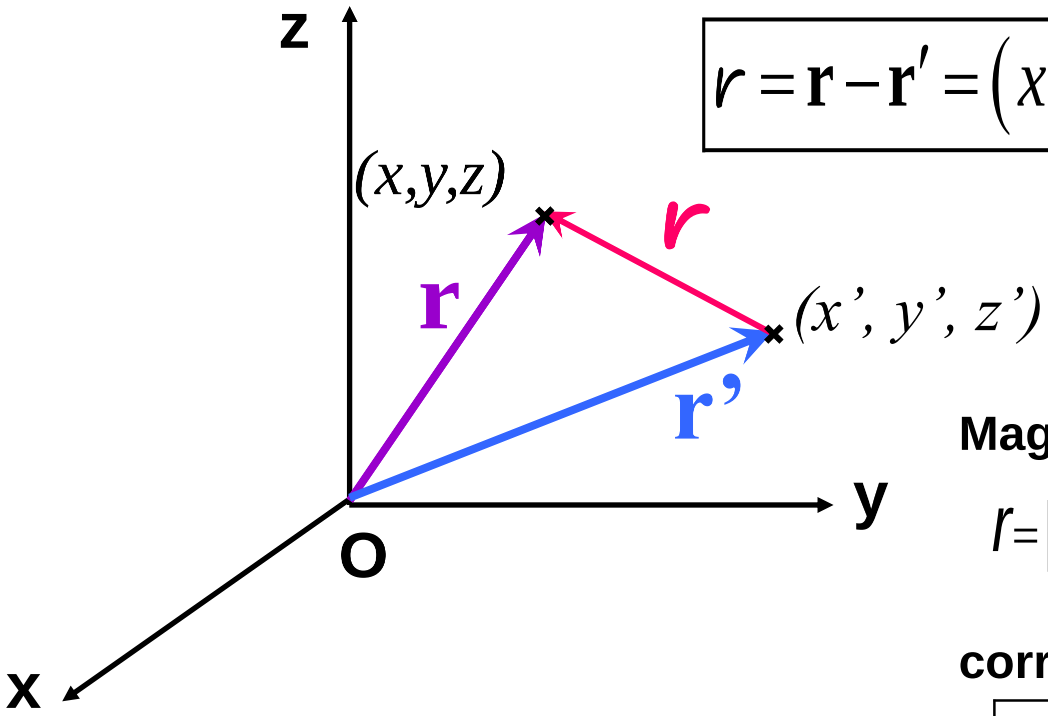
$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$

(b) Infinitesimal displacement vector pointing from (x, y, z) to $(x+dx, y+dy, z+dz)$:

$$d\mathbf{l} = dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}}$$



(c) Separation vector from **r'** *to* **r** *:*



$$\mathscr{r} = \mathbf{r} - \mathbf{r}' = (x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}}$$

Magnitude:

$$\mathscr{r} = |\mathbf{r} - \mathbf{r}'| = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

corresponding unit vector:

$$\hat{\mathscr{r}} = \frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}} = \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$$

Vector Analysis

Vector Transformation

How vectors transform?

Let R be the rotation matrix:

$$R = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix}$$

$\mathbf{A} = \left(A_x, A_y, A_z\right)$ is a vector if, under the rotation, its components transform like

$$\bar{A}_i = \sum_{j=1}^{3} R_{ij} A_j$$

where $\bar{A}_i$ are the components of the vector in the rotated frame, and $1 \leftrightarrow x, 2 \leftrightarrow y, 3 \leftrightarrow z$

In matrix notation, the transformation reads

$$\begin{pmatrix} \bar{A}_x \\ \bar{A}_y \\ \bar{A}_z \end{pmatrix} = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

Note:

Rotation matrices have the special property that $R^{-1} = R^T$ *(they are orthogonal).*

Tensors

- Scalar = 0th rank tensor
- Vector = 1st rank tensor
- 2nd rank tensor → In the form of a matrix
- In general, tensor can be of any rank

A (second rank) tensor

$$T = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$$

has nine components, which transform under rotation like

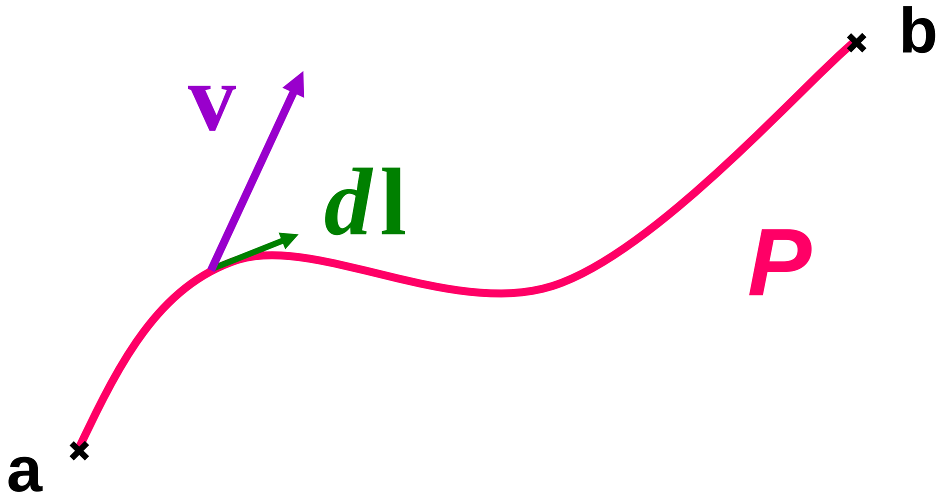
$$\bar{T}_{ij} = \sum_{k=1}^{3} \sum_{l=1}^{3} R_{ik} R_{jl} T_{kl}$$

Vector Calculus

Line, Surface, and
Volume Integrals

(a) Line Integrals

$$\int_{\mathbf{a}P}^{\mathbf{b}} \mathbf{v} \cdot d\mathbf{l}$$



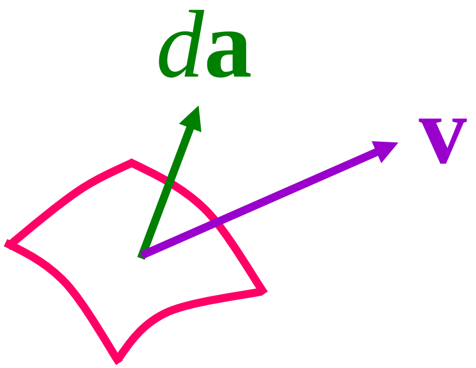
For closed path, **a**=**b**, we use the notation

$$\oint \mathbf{v} \cdot d\mathbf{l}$$

- In general, the line integral depends on **a**, **b**, and the path *P*.
- In particular, if it depends only on **a**, **b**, and is independent of *P*, we say that the field is a ***conservative*** field.

(b) Surface Integral

$$\int_S \mathbf{v} \cdot d\mathbf{a}$$



da: Area of the infinitesimal area
*Direction of d**a**: Normal to the area*
(two directions, choose either one)

For closed surface, we use the notation

$$\oint \mathbf{v} \cdot d\mathbf{a}$$

For closed surface, usually we take ***outward*** as the direction of $d\mathbf{a}$.

(c) Volume Integral

(i) Volume integral of scalar field T

$$\int_V T\,d\tau \qquad\qquad d\tau = dxdydz$$

(ii) Volume integral of vector field **v**

$$\int_V \mathbf{v}\,d\tau$$

$$\int_V \mathbf{v}\,d\tau = \int_V (v_x\hat{x} + v_y\hat{y} + v_z\hat{z})d\tau = \hat{x}\int v_x d\tau + \hat{y}\int v_y d\tau + \hat{z}\int v_z d\tau$$

Vector Calculus

The “Del” Operator

What is the “Del” Operator?

Definition of the “del” operator:

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

What is the "Del" Operator?

- It is an operator, without specific meaning until a function is provided for it to act on

- It is a vector operator. Though not being a vector in the usual sense, it mimics the behavior of an ordinary vector in virtually every way

  1. *Acting on a scalar field T: the* ***gradient***
  2. *Acting on a vector field* ***v****, via the dot product: the* ***divergence***
  3. *Acting on a vector field* ***v****, via the cross product: the* ***curl***

Vector Calculus

Gradient, Divergence, and Curl

Gradient

Consider a scalar field *T*(*x*,*y*,*z*)

$$dT = \left(\frac{\partial T}{\partial x}\right)dx + \left(\frac{\partial T}{\partial y}\right)dy + \left(\frac{\partial T}{\partial z}\right)dz$$

It can be written as

$$dT = \left(\frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}\right)\cdot\left(dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}}\right)$$

Recall that

$$d\mathbf{l} = dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}}$$

Then

$$\boxed{dT = \left(\nabla T\right)\cdot\left(d\mathbf{l}\right)}$$

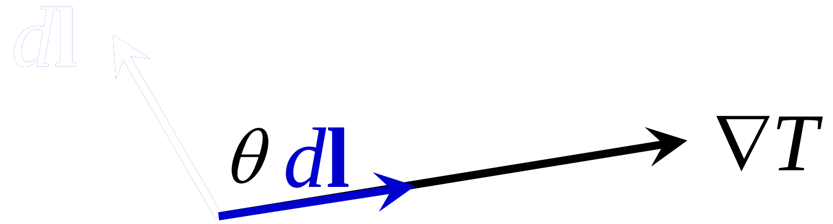
Geometrical meaning:

$$dT = |\nabla T||d\mathbf{l}|\cos\theta$$

where θ is the angle between ∇T and $d\mathbf{l}$, the direction along which you move

When $d\mathbf{l}$ points to the same direction as ∇T ,

$\theta = 0$ and $dT = |\nabla T||d\mathbf{l}|$ attains its maximum value.



- The gradient ∇T points in the direction of maximum increase of *T*
- The magnitude $|\nabla T|$ gives the slope (rate of increase) along this maximal direction

Example:

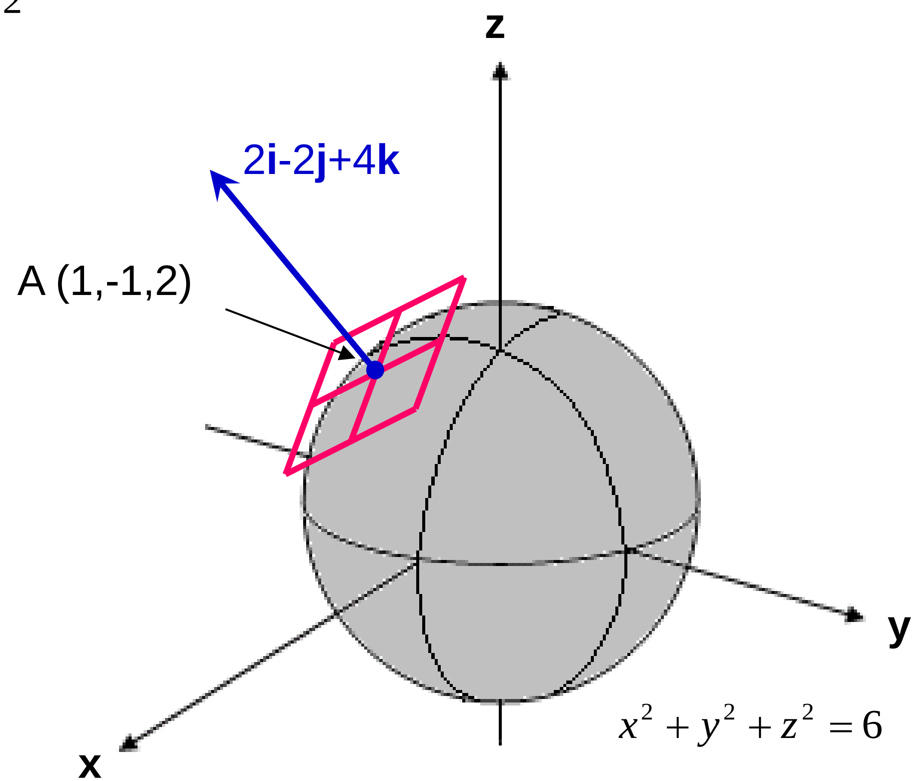
$$T(x, y, z) = x^2 + y^2 + z^2 = r^2$$

$$\nabla T = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$= 2\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= 2\vec{r}$$

$$\left|\nabla T\right| = 2r$$



Consider point A(1,-1,2),

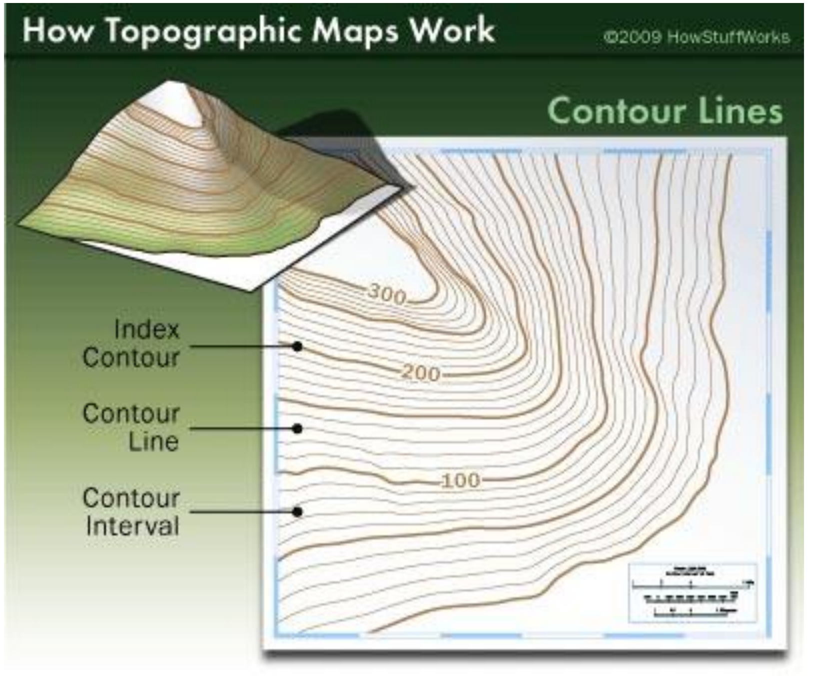
Max. rate of increase of T

=|∇T(1,-1,2)|= $2\sqrt{6}$

In particular, when

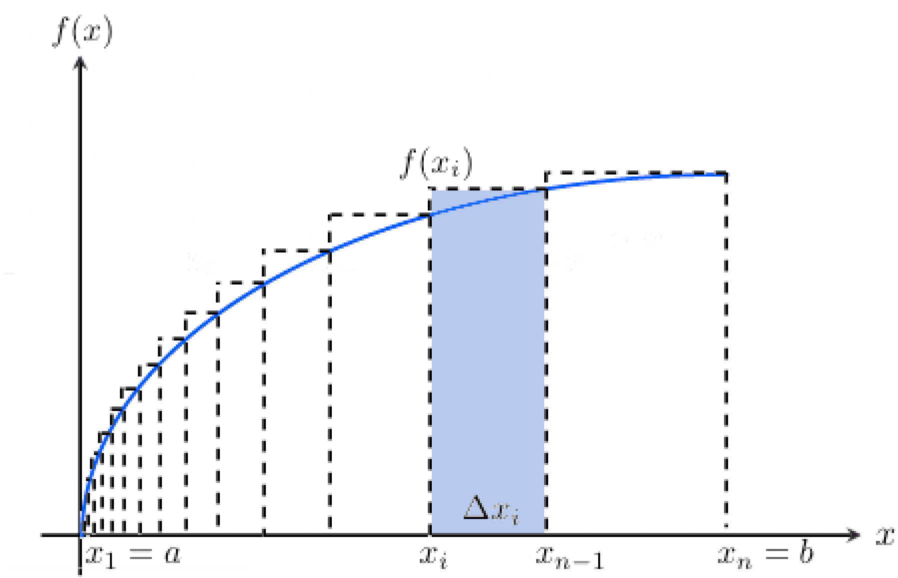
$$d\mathbf{l} \perp \nabla T$$

***dT*=0 → moving along the contour line**



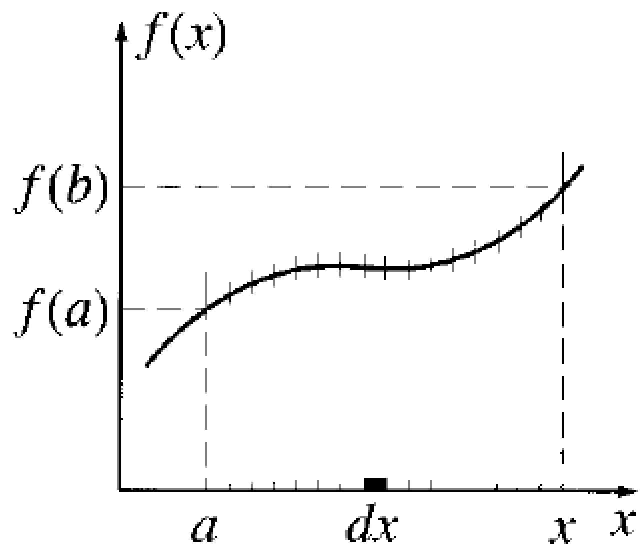
The fundamental theorem of Calculus

$$\int_b^a f'(x)\,dx = f(a) - f(b)$$



Area under curve $f(x) = \int_a^b f(x)dx = F(b) - F(a) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i$

(method of exhaustion)

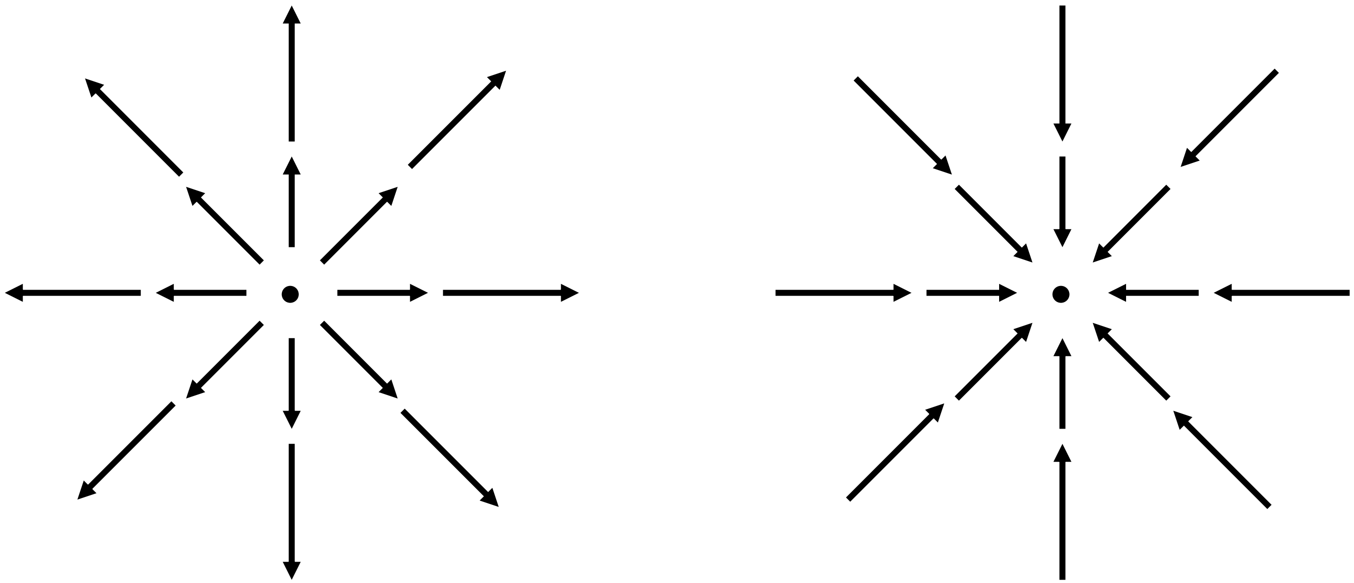


The fundamental theorem of gradient:

$$\int_{\mathbf{a}P}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = \int_{\mathbf{a}P}^{\mathbf{b}} dT = T(\mathbf{b}) - T(\mathbf{a})$$

Divergence

- To measure the flux (“source/sink”) density

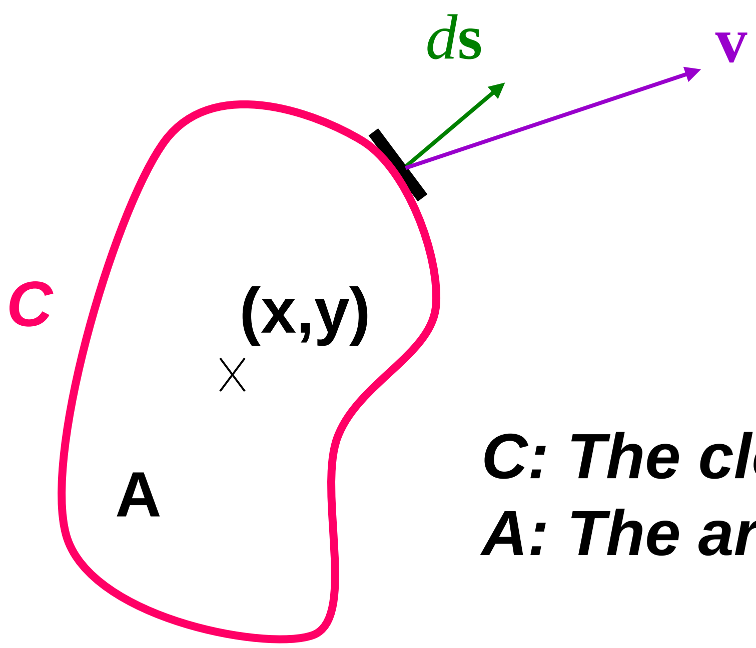


SOURCE **SINK**

How much does a vector spreads out (diverges) from a certain point

Divergence for a flat plane: 2D

A vector field v in the *xy* plane. Evaluate the flux through a closed curve enclosing a point (*x*,*y*)



The integral is

$$\oint_C \mathbf{v} \cdot d\mathbf{s}$$

C: The closed curve

A: The area enclosed by the loop

This integral measures the flux and hence the source enclosed by the loop.
Therefore,

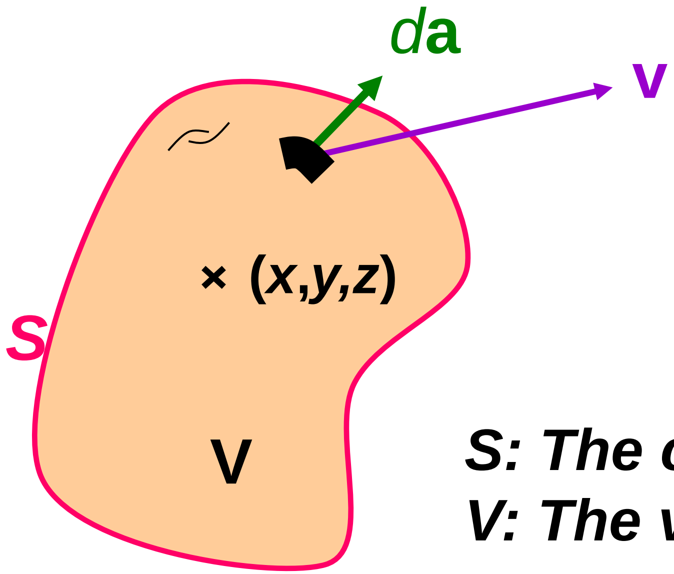
$$\lim_{A\to 0}\frac{1}{A}\oint_C \mathbf{v}\cdot d\mathbf{s}$$

is the flux density at that point.

In 2D, the divergence of a vector field v is defined by

$$\nabla\cdot\mathbf{v}=\lim_{A\to 0}\frac{1}{A}\oint_C \mathbf{v}\cdot d\mathbf{s}$$

Divergence for a closed surface in 3D:



$$\nabla \cdot \mathbf{v} = \lim_{V \to 0} \frac{1}{V} \oint_S \mathbf{v} \cdot d\mathbf{a}$$

S: The closed surface
V: The volume enclosed by the surface

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

The Divergence Theorem

(Gauss' or Green's Theorem)

Consider the integral:

$$\oint_S \mathbf{v} \cdot d\mathbf{a}$$

over a closed surface *S* enclosing a volume *V*

The fundamental theorem for divergences states that:

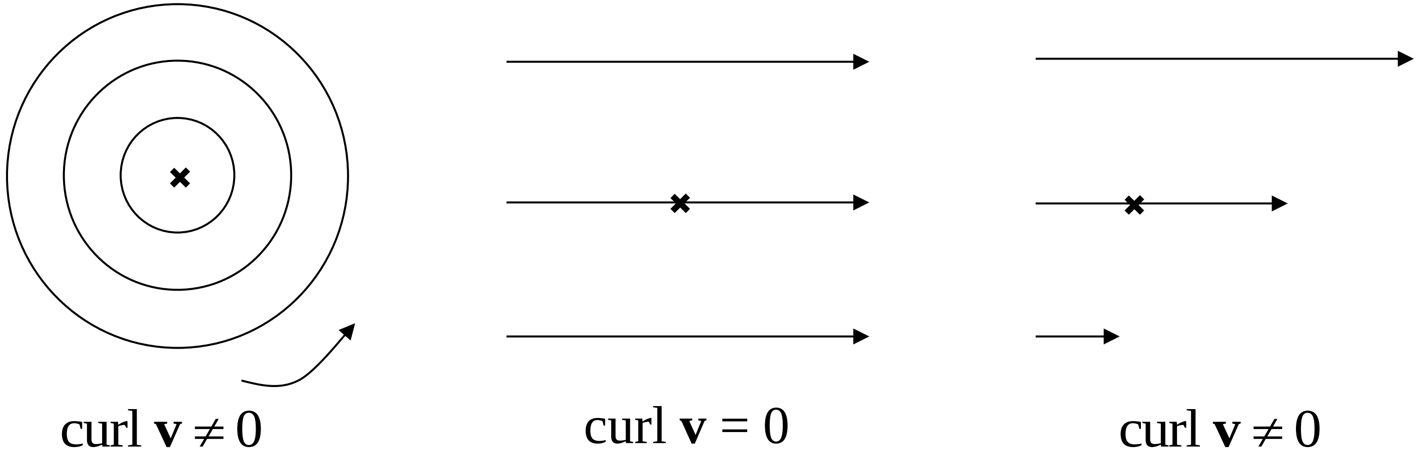
$$\boxed{\int_{\mathcal{V}} (\nabla \cdot \mathbf{v}) d\tau = \oint_S \mathbf{v} \cdot d\mathbf{a}}$$

Integral of a divergence over a volume is equal to the value of the function at the boundary!

Curl

The curl of a vector field v measures how much the vector v "curls around" the point in question.

You can depict it as placing a tiny paddle wheel probe in a flowing fluid to determine whether it will turn and how fast the angular velocity is



This depends on the *orientation* (the plane) of the paddle wheel.

Consider a plane with normal vector n.
Draw a small closed loop *C* on the plane around the point in question, enclosing an area *A*.

Evaluate

$$\lim_{A \to 0} \frac{1}{A} \oint_C \mathbf{v} \cdot d\mathbf{l}$$

The *direction* of the line integral is determined by *right-hand rule*

Curl v is a ***vector*** defined by:

$$\nabla \times \mathbf{v} = \hat{\mathbf{x}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

$$= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) \times \left(v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}\right)$$

In general, for a plane with unit normal vector n

$$\lim_{A\to 0}\frac{1}{A}\oint_C \mathbf{v}\cdot d\mathbf{l} = (\nabla\times\mathbf{v})\cdot\mathbf{n}$$

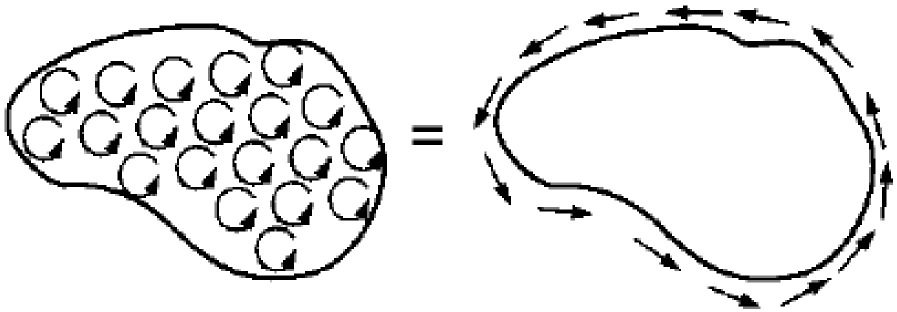
Physical meaning: curl v is a vector pointing along a direction in which a paddle wheel will have the *greatest tendency to turn*

For a small loop *C* with area *da* and unit normal vector n,

$$\oint_{C'} \mathbf{v}\cdot d\mathbf{l} \approx (\nabla\times\mathbf{v})\cdot\mathbf{n}\,da = (\nabla\times\mathbf{v})\cdot d\mathbf{a}$$

Stokes' theorem

Consider the line integral along a closed loop *C*



$$\oint_S (\nabla \times \mathbf{v})\, d\mathbf{a} = \oint_C \mathbf{v} \cdot d\mathbf{l}$$

Integral of a curl over a surface is equal to the value of the function at the boundary

Vector Calculus

Product Rules Involving the Del Operator

Product Rules Involving the Del Operator

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

$$(\mathbf{A}\cdot\nabla)\mathbf{B} = \left(A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z}\right)\left(B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}\right)$$

Note:

$$= \hat{\mathbf{x}}\left(A_x\frac{\partial B_x}{\partial x} + A_y\frac{\partial B_x}{\partial y} + A_z\frac{\partial B_x}{\partial z}\right)$$

$$+ \hat{\mathbf{y}}\left(A_x\frac{\partial B_y}{\partial x} + A_y\frac{\partial B_y}{\partial y} + A_z\frac{\partial B_y}{\partial z}\right)$$

$$+ \hat{\mathbf{z}}\left(A_x\frac{\partial B_z}{\partial x} + A_y\frac{\partial B_z}{\partial y} + A_z\frac{\partial B_z}{\partial z}\right)$$

Product Rules Involving the Del Operator

- $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$
- $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$
- $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$
- $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Vector Calculus

Second Derivatives

Second Derivatives

- *Laplacian – Divergence of gradient*

$$\nabla^2 T = \nabla \cdot (\nabla T)$$

$$= \left(\hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \right) \cdot \left(\hat{\mathbf{x}} \frac{\partial T}{\partial x} + \hat{\mathbf{y}} \frac{\partial T}{\partial y} + \hat{\mathbf{z}} \frac{\partial T}{\partial z} \right)$$

$$= \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

The Laplacian of a scalar *T* is a scalar.
The Laplacian of a vector **v** is similarly defined by:

$$\nabla^2 \mathbf{v} = \hat{\mathbf{x}} \nabla^2 v_x + \hat{\mathbf{y}} \nabla^2 v_y + \hat{\mathbf{z}} \nabla^2 v_z$$

Second Derivatives

- ***Curl of gradient***

  The curl of a gradient is always zero

$$\nabla \times \left(\nabla T \right) = 0$$

- ***Gradient of divergence***

$$\nabla \left(\nabla \cdot \mathbf{v} \right)$$

(Seldom occurs in physics)

Second Derivatives

- ***Divergence of curl***

  The divergence of a curl is always zero

$$\nabla \cdot (\nabla \times \mathbf{v}) = 0$$

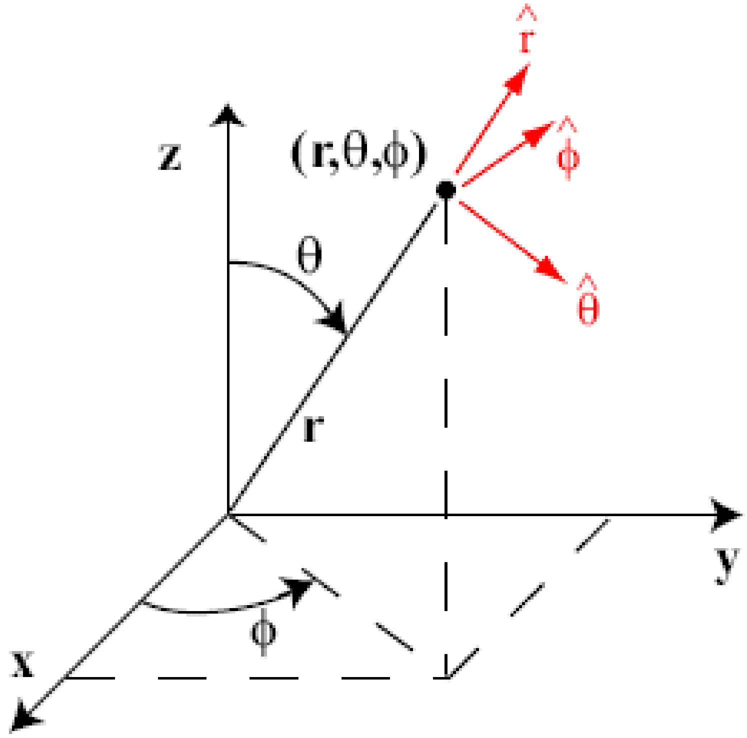
- ***Curl of Curl***

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla (\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}$$

Curvilinear Coordinates

Spherical Polar Coordinates

Spherical Polar Coordinates (r,θ,ϕ)



r: Distance from the origin

θ: Polar angle – angle down from the z axis

ϕ: Azimuthal angle – angle around from the x axis (right-hand rule)

Relation between Cartesian and Spherical Polar Coordinates:

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \Leftrightarrow \begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \cos^{-1}\dfrac{z}{\sqrt{x^2 + y^2 + z^2}} \\ \phi = \tan^{-1}\dfrac{y}{x} \end{cases}$$

Basis vectors

$$\begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\hat{\mathbf{x}} + \cos\theta\sin\phi\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}} \\ \hat{\boldsymbol{\varphi}} = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}} \end{cases}$$

$$\Leftrightarrow \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\hat{\mathbf{r}} + \cos\theta\cos\phi\hat{\boldsymbol{\theta}} - \sin\phi\hat{\boldsymbol{\varphi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\hat{\mathbf{r}} + \cos\theta\sin\phi\hat{\boldsymbol{\theta}} + \cos\phi\hat{\boldsymbol{\varphi}} \\ \hat{\mathbf{z}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}} \end{cases}$$

Basis vectors

- *The direction of the basis vectors is along the direction of increase of the corresponding coordinates, keeping others fixed.*
- *They constitute an orthogonal basis set*

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\varphi}} \cdot \hat{\boldsymbol{\varphi}} = 1$$

$$\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\theta}} = \hat{\mathbf{r}} \cdot \hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\varphi}} = 0$$

$$\hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\varphi}} = \hat{\mathbf{r}}$$

$$\hat{\boldsymbol{\varphi}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\theta}}$$

$$\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\varphi}}$$

Basis vectors

- *A vector **A** can be expressed in component form as*

$$\mathbf{A} = A_r\hat{\mathbf{r}} + A_\theta\hat{\boldsymbol{\theta}} + A_\phi\hat{\boldsymbol{\varphi}}$$

- *The same basis vector associated with different points are along different directions*

- $\dfrac{\partial\hat{\mathbf{r}}}{\partial r} = \dfrac{\partial\hat{\boldsymbol{\theta}}}{\partial r} = \dfrac{\partial\hat{\boldsymbol{\varphi}}}{\partial r} = \dfrac{\partial\hat{\boldsymbol{\varphi}}}{\partial\theta} = \mathbf{0}$
- $\dfrac{\partial\hat{\mathbf{r}}}{\partial\theta} = \hat{\boldsymbol{\theta}}$
- $\dfrac{\partial\hat{\mathbf{r}}}{\partial\phi} = \sin\theta\hat{\boldsymbol{\varphi}}$
- $\dfrac{\partial\hat{\boldsymbol{\theta}}}{\partial\theta} = -\hat{\mathbf{r}}$
- $\dfrac{\partial\hat{\boldsymbol{\theta}}}{\partial\phi} = \cos\theta\hat{\boldsymbol{\varphi}}$
- $\dfrac{\partial\hat{\boldsymbol{\varphi}}}{\partial\phi} = -\sin\theta\hat{\mathbf{r}} - \cos\theta\hat{\boldsymbol{\theta}}$

Infinitesimal displacement

- $d\mathbf{l} = dr\hat{\mathbf{r}} + rd\theta\hat{\boldsymbol{\theta}} + r\sin\theta d\phi\hat{\boldsymbol{\varphi}}$

Infinitesimal volume element

- $d\tau = r^2 \sin\theta dr d\theta d\phi$

Infinitesimal area element on a spherical surface with radius *r*

- $d\mathbf{a} = r^2 \sin\theta d\theta d\phi \hat{\mathbf{r}}$

The gradient, divergence, curl, Laplacian etc. in spherical polar coordinate can be derived by the relations between the coordinates and the basis vectors.

For example:

$$\nabla T = \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}$$

$$= \left(\frac{\partial T}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial T}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial T}{\partial \phi}\frac{\partial \phi}{\partial x} \right)\left(\sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\varphi}} \right) + \cdots$$

The expressions are:

- *Gradient*

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\varphi}}$$

- *Divergence*

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

- *Curl*

$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}\left(\sin\theta v_\phi\right) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}\left(r v_\phi\right)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r v_\theta\right) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\varphi}}$$

Hence,

- *Laplacian*

$$\nabla^2 T = \nabla \cdot (\nabla T)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 (\nabla T)_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta (\nabla T)_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial(\nabla T)_\phi}{\partial\phi}$$

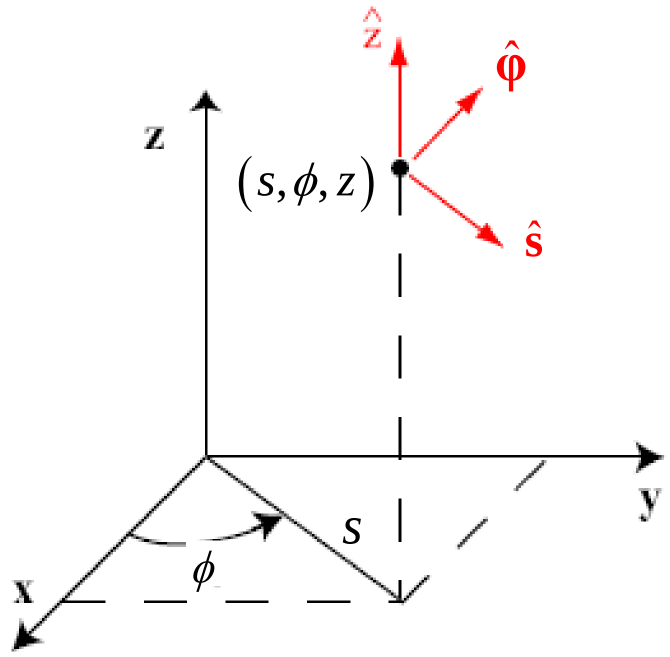
$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{1}{r}\frac{\partial T}{\partial\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{1}{r\sin\theta}\frac{\partial T}{\partial\phi}\right)$$

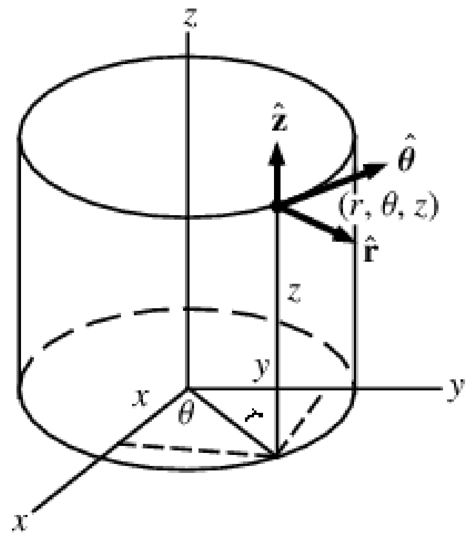
$$\boxed{\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}}$$

Curvilinear Coordinates

Cylindrical Coordinates

Cylindrical Coordinates (s, ϕ, z)





s: Distance from the z axis

ϕ: *Angle around from the x axis (right-hand rule)*

z: Distance from the x-y plane

Relation between Cartesian and Cylindrical Coordinates:

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \Leftrightarrow \begin{cases} s = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}\dfrac{y}{x} \\ z = z \end{cases}$$

Basis vectors

$$\begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\varphi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\Leftrightarrow \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\varphi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\varphi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

Basis vectors

- *The direction of the basis vectors is along the direction of increase of the corresponding coordinates, keeping others fixed.*
- *They constitute an orthogonal basis set*

$$\hat{\mathbf{s}} \cdot \hat{\mathbf{s}} = \hat{\boldsymbol{\varphi}} \cdot \hat{\boldsymbol{\varphi}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1$$

$$\hat{\mathbf{s}} \cdot \hat{\boldsymbol{\varphi}} = \hat{\mathbf{s}} \cdot \hat{\mathbf{z}} = \hat{\boldsymbol{\varphi}} \cdot \hat{\mathbf{z}} = 0$$

$$\hat{\mathbf{s}} \times \hat{\boldsymbol{\varphi}} = \hat{\mathbf{z}}$$

$$\hat{\boldsymbol{\varphi}} \times \hat{\mathbf{z}} = \hat{\mathbf{s}}$$

$$\hat{\mathbf{z}} \times \hat{\mathbf{s}} = \hat{\boldsymbol{\varphi}}$$

Basis vectors

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- $\dfrac{\partial\hat{\mathbf{z}}}{\partial s} = \dfrac{\partial\hat{\mathbf{z}}}{\partial\phi} = \dfrac{\partial\hat{\mathbf{z}}}{\partial z} = \dfrac{\partial\hat{\boldsymbol{\varphi}}}{\partial s} = \dfrac{\partial\hat{\boldsymbol{\varphi}}}{\partial z} = \dfrac{\partial\hat{\mathbf{s}}}{\partial s} = \dfrac{\partial\hat{\mathbf{s}}}{\partial z} = \mathbf{0}$
- $\dfrac{\partial\hat{\mathbf{s}}}{\partial\phi} = \hat{\boldsymbol{\varphi}}$
- $\dfrac{\partial\hat{\boldsymbol{\varphi}}}{\partial\phi} = -\hat{\mathbf{s}}$

Infinitesimal displacement

- $d\mathbf{l} = ds\hat{\mathbf{s}} + sd\phi\hat{\boldsymbol{\varphi}} + dz\hat{\mathbf{z}}$

Infinitesimal volume element

- $d\tau = sdsd\phi dz$

Infinitesimal area element on a spherical surface with radius *r*

- $d\mathbf{a} = sd\phi dz\hat{\mathbf{s}}$

The gradient, divergence, curl, Laplacian etc. in spherical polar coordinate can be derived by the relations between the coordinates and the basis vectors.

For example:

$$\nabla T = \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}$$

$$= \left(\frac{\partial T}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial T}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial T}{\partial z}\frac{\partial z}{\partial x} \right)\left(\cos\phi\hat{\mathbf{s}} - \sin\phi\hat{\boldsymbol{\varphi}}\right) + \cdots$$

The expressions are:

- *Gradient*

$$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\varphi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}$$

- *Divergence*

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}\left(s v_s\right) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

- *Curl*

$$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\varphi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}\left(s v_\phi\right) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Hence,

- *Laplacian*

$$\nabla^2 T = \nabla \cdot \left(\nabla T \right)$$

$$= \frac{1}{s}\frac{\partial}{\partial s}\left(s\left(\nabla T \right)_s \right) + \frac{1}{s}\frac{\partial \left(\nabla T \right)_\phi}{\partial \phi} + \frac{\partial \left(\nabla T \right)_z}{\partial z}$$

$$= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s} \right) + \frac{1}{s}\frac{\partial}{\partial \phi}\left(\frac{1}{s}\frac{\partial T}{\partial \phi} \right) + \frac{\partial}{\partial z}\frac{\partial T}{\partial z}$$

$$\boxed{\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s} \right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}}$$

Dirac Delta Function

Consider the vector field

$$\mathbf{v} = \frac{1}{r^2}\hat{\mathbf{r}}$$

Using divergence in spherical polar coordinate

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = 0 \qquad \textbf{\textit{for}} \quad r \neq 0$$

For r = 0, consider the flux through a spherical surface S with radius R centered at the origin

$$\oint_S \mathbf{v} \cdot d\mathbf{a} = \int \left(\frac{1}{R^2} \hat{\mathbf{r}} \right) \cdot \left(R^2 \sin\theta d\theta d\phi \hat{\mathbf{r}} \right)$$

$$= \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi = 4\pi$$

Therefore, $\nabla \cdot \mathbf{v} = \lim_{V \to 0} \frac{1}{V} \oint_S \mathbf{v} \cdot d\mathbf{a}$

$$= \lim_{R \to 0} \frac{4\pi}{4\pi R^3 / 3} = \lim_{R \to 0} \frac{3}{R^3} = \infty$$

In addition,

$$\int_V \nabla \cdot \mathbf{v} d\tau = 0$$

if V does not include the origin.

If V includes the origin, consider the spherical surface S with radius R centered at the origin. From divergence theorem

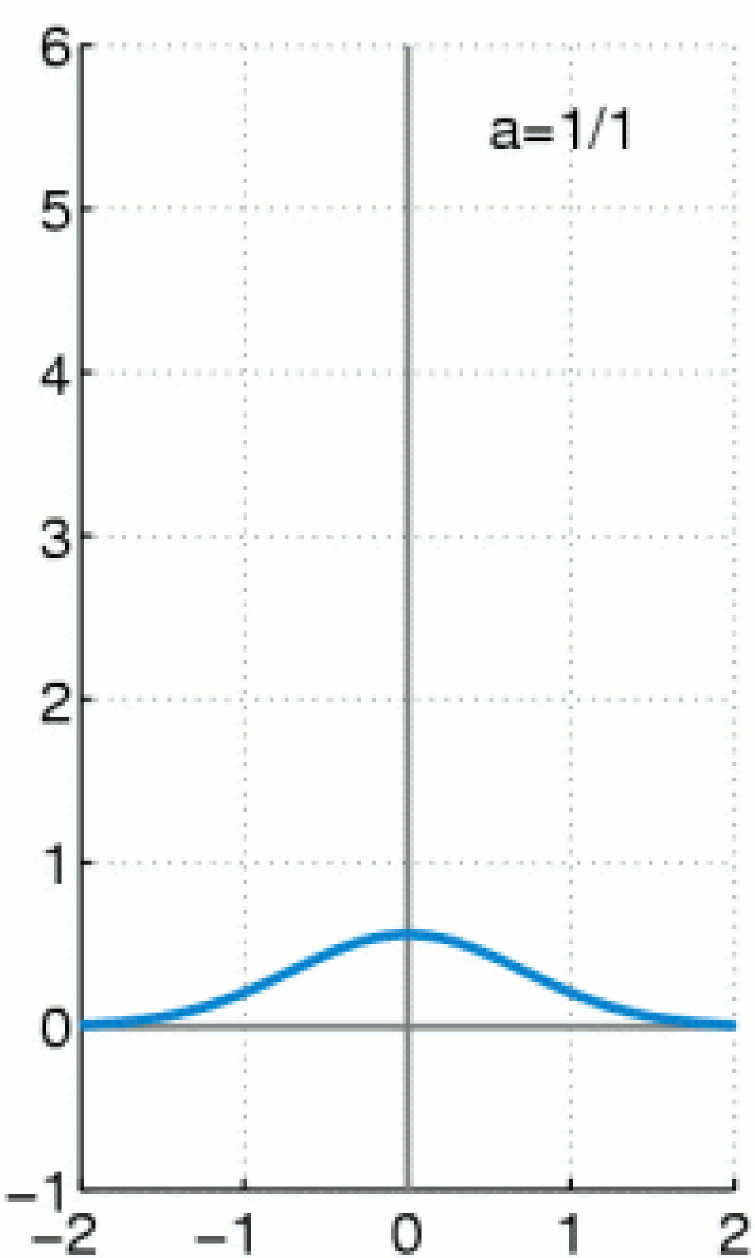
$$\int_V \nabla \cdot \mathbf{v} d\tau = \oint_S \mathbf{v} \cdot d\mathbf{a} = 4\pi$$

What is a delta function?

The delta function is a generalized function, or distribution, that can be thought of as the limit of a class of delta sequences.

The delta function is sometimes called "Dirac delta function" or the "impulse symbol"

$$\delta_a(x) = \lim_{a \to 0^+} \frac{1}{a\sqrt{\pi}} e^{-x^2/a^2}$$



Dirac Delta Function

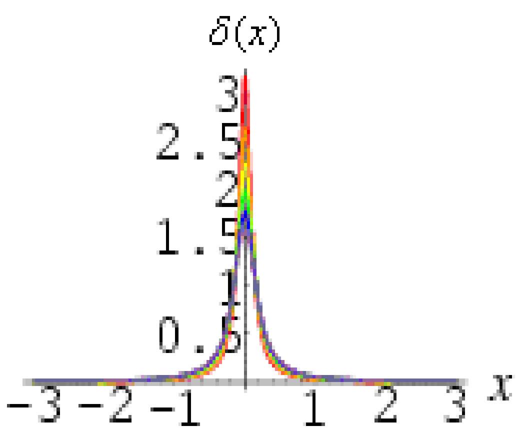
One-Dimensional

A one-dimensional delta function $\delta(x)$ satisfies

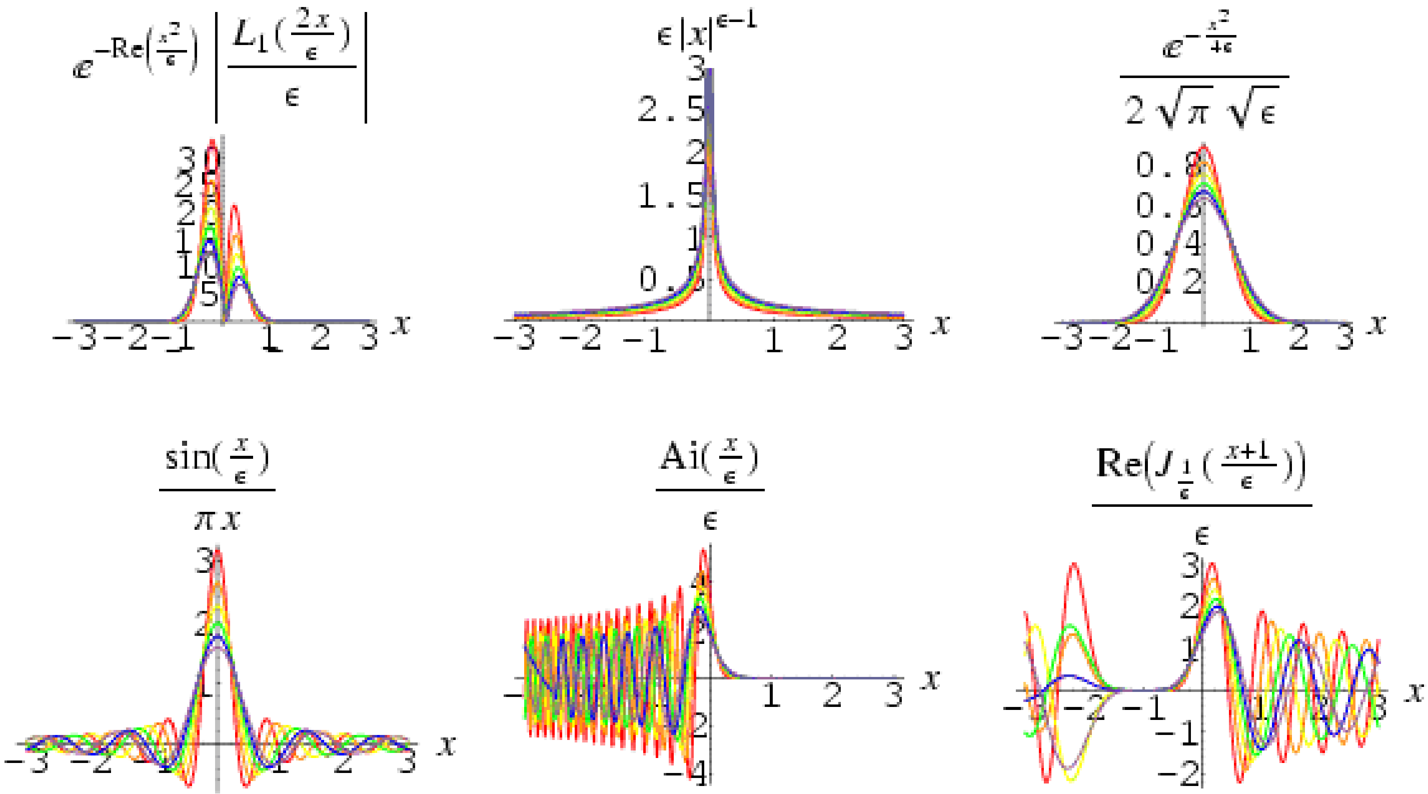
$$\int_a^b \delta(x)\,dx = 0 \qquad \text{,if } 0 \notin (a,b)$$

$$\int_a^b \delta(x)\,dx = 1 \qquad \text{,if } 0 \in (a,b)$$

e.g. $\delta(x) = \dfrac{\epsilon}{\pi\left(x^2 + \epsilon^2\right)}$



Other examples:



We get $\delta(x)$ *when* $\varepsilon \to 0$ *in the above examples.*

For "well-behaved" functions f(x):

$$\int_a^b f(x)\delta(x)dx = 0 \qquad \textbf{\textit{,if}}\ \ 0 \notin (a,b)$$

$$\int_a^b f(x)\delta(x)dx = f(0) \quad \textbf{\textit{,if}}\ \ 0 \in (a,b)$$

The definition can be generalized to $\delta\left(x-x_0\right)$:

$$\int_a^b \delta(x-x_0)dx = 0 \quad \text{,if } x_0 \notin (a,b)$$

$$\int_a^b \delta(x-x_0)dx = 1 \quad \text{,if } x_0 \in (a,b)$$

For "well-behaved" functions f(x):

$$\int_a^b f(x)\delta(x-x_0)dx = 0 \quad \text{,if } x_0 \notin (a,b)$$

$$\int_a^b f(x)\delta(x-x_0)dx = f(x_0) \quad \text{,if } x_0 \in (a,b)$$

Dirac Delta Function

Three-Dimensional

The 3D delta function $\delta^3(\mathbf{r})$ satisfies the basic requirement that

$$\int_V \delta^3(\mathbf{r})\,d\tau = 0 \qquad \textit{,if } \mathbf{0} \notin V$$

$$\int_V \delta^3(\mathbf{r})\,d\tau = 1 \qquad \textit{,if } \mathbf{0} \in V$$

and,

$$\int_V f(\mathbf{r})\,\delta^3(\mathbf{r})\,d\tau = 0 \qquad \textit{,if } \mathbf{0} \notin V$$

$$\int_V f(\mathbf{r})\,\delta^3(\mathbf{r})\,d\tau = f(\mathbf{0}) \qquad \textit{,if } \mathbf{0} \in V$$

The definition can be generalized to $\delta(\mathbf{r}-\mathbf{r_0})$***:***

$$\int_V \delta(\mathbf{r}-\mathbf{r_0})\,d\tau = 0 \qquad \text{,if } \mathbf{r_0} \notin V$$

$$\int_V \delta(\mathbf{r}-\mathbf{r_0})\,d\tau = 1 \qquad \text{,if } \mathbf{r_0} \in V$$

For "well-behaved" functions f(x):

$$\int_V f(\mathbf{r})\delta(\mathbf{r}-\mathbf{r_0})\,d\tau = 0 \qquad \text{,if } \mathbf{r_0} \notin V$$

$$\int_V f(\mathbf{r})\delta(\mathbf{r}-\mathbf{r_0})\,d\tau = f(\mathbf{r_0}) \quad \text{,if } \mathbf{r_0} \in V$$

We have shown that

$$\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2} \right) = 4\pi\delta^3(\mathbf{r})$$

Or more generally,

$$\boxed{\nabla \cdot \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) = 4\pi\delta^3(\mathscr{r})}$$

where $\mathscr{r} = \mathbf{r} - \mathbf{r}'$

Note: r' is kept constant during differentiation

It can be shown that

$$\nabla\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}$$

Therefore,

$$\nabla^2 \frac{1}{r} = -4\pi\delta^3(r)$$

THE END