PHYS 3038 Optics
L21 Fourier Optics
Reading Material: Ch11

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11.1 Introduction: Why Fourier Optics

- Example: Free space –superposition principle

$$E(x,y,z,t) = f(x,y,z)e^{-i\omega t}$$

$$k = \frac{\omega}{c} = \sqrt{k_x^2 + k_y^2 + k_z^2}$$

$$k_z = \sqrt{k^2 - k_x^2 - k_y^2}$$

$$f(x,y,z) = \iint F(k_x,k_y)e^{i[k_x x + k_y y + \sqrt{k^2 - k_x^2 - k_y^2} z]} dk_x dk_y$$

$$= \iint F(k_x,k_y)\, e^{i\sqrt{k^2 - k_x^2 - k_y^2} z} e^{i[k_x x + k_y y]} dk_x dk_y$$

$$= \iint F(k_x,k_y)\, H(k_x,k_y,z) e^{i[k_x x + k_y y]} dk_x dk_y$$

11.2.1 1D Fourier Transform

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(k) e^{ikx} dk = \mathcal{F}^{-1}\{F(k)\}$$

$$= \mathcal{F}^{-1}\{\mathcal{F}\{f(x)\}\}$$

$$F(k) = \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx = \mathcal{F}\{f(x)\}$$

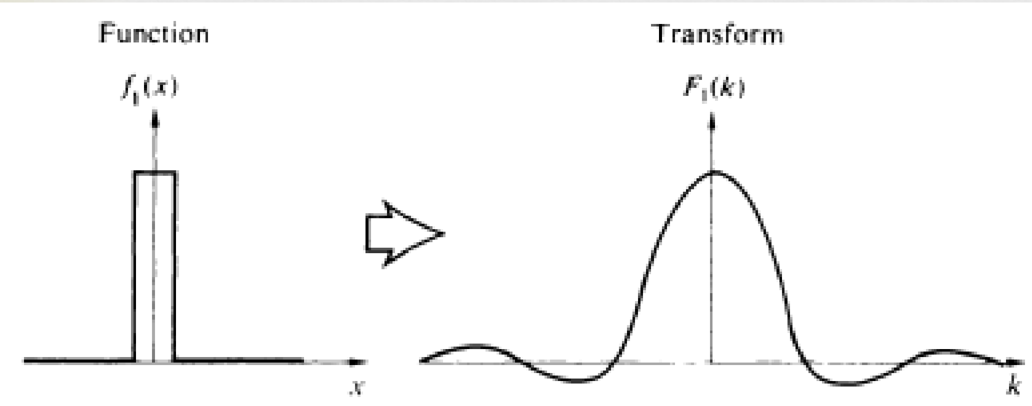
$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx} dk = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ikx} dk$$

$$\mathcal{F}\{\delta(x)\} = 1$$

$$f(x) = 1$$

$$\mathcal{F}\{1\} = \int_{-\infty}^{+\infty} e^{-ikx} dx = 2\pi\delta(k)$$

FT of a rectangular function



$$f(x) = \begin{cases} 1 & \frac{a}{2} < x < \frac{a}{2} \\ 0 & others \end{cases}$$

$$F(k) = \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{-ikx} dx = \left[\frac{e^{-ikx}}{-ik}\right]_{-\frac{a}{2}}^{\frac{a}{2}} = \frac{e^{-ika/2} - e^{ika/2}}{-ik}$$

$$= \frac{-2i \sin\frac{ka}{2}}{-ik} = a\,\frac{\sin\frac{ka}{2}}{\frac{ka}{2}} = a\,\mathrm{sinc}\,\frac{ka}{2}$$

FT of a Triangle Function

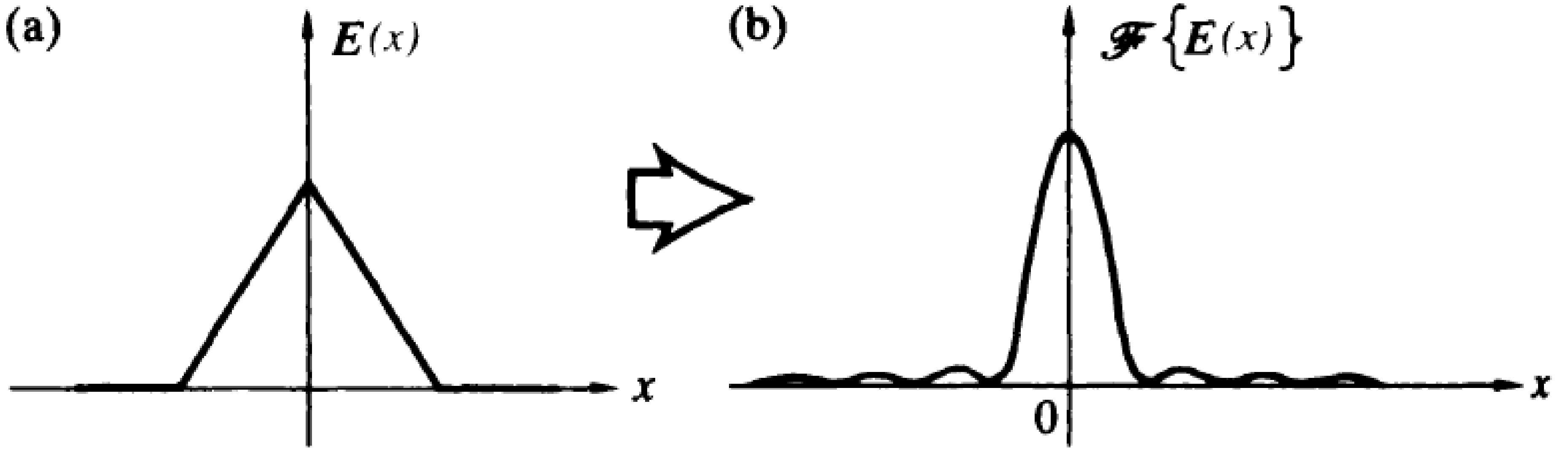


Figure 11.6 The transform of the triangle function is the sinc^2 function.

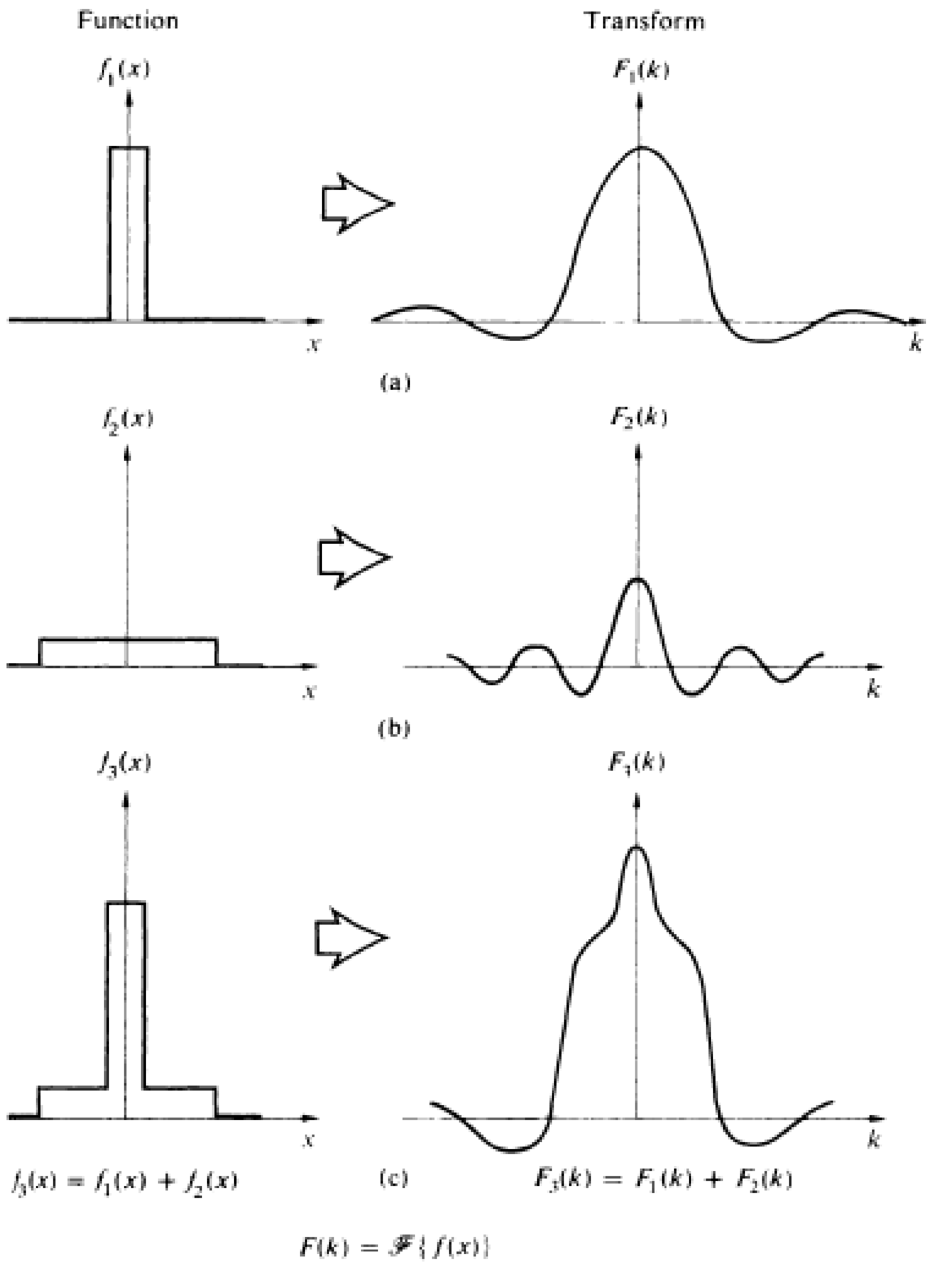


Figure 11.1 A composite function and its Fourier transform.

FT of the Gaussian Function

$$\int_{-\infty}^{+\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

$$f(x) = Ce^{-ax^2}$$

$$F(k) = \int_{-\infty}^{+\infty} Ce^{-ax^2} e^{-ikx} dx = \int_{-\infty}^{+\infty} Ce^{-a(x^2 + i\frac{k}{a}x)} dx$$

$$= Ce^{-\frac{k^2}{4a}} \int_{-\infty}^{+\infty} e^{-a(x^2 + i\frac{k}{a}x - \frac{k^2}{4a^2})} dx = Ce^{-\frac{k^2}{4a}} \int^{+\infty} e^{-a(x + i\frac{k}{2a})^2} dx$$

$$= C\sqrt{\frac{\pi}{a}} e^{-\frac{k^2}{4a}}$$

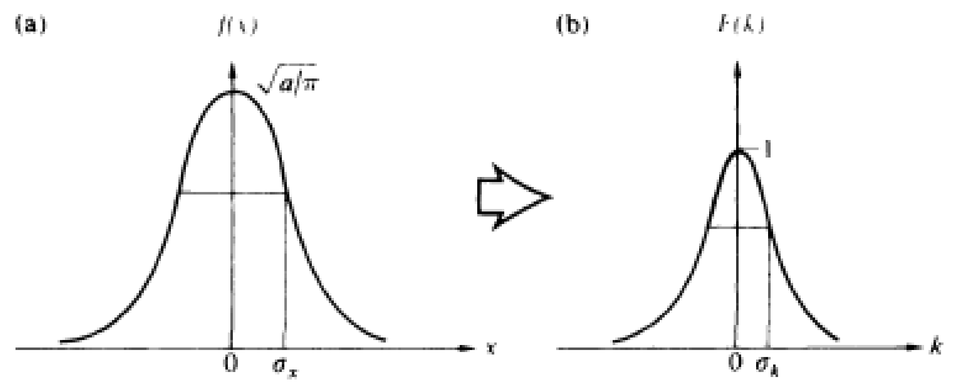


Figure 11.2 A Gaussian and its Fourier transform.

$$F(k) = \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx = \mathcal{F}\{f(x)\}$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(k) e^{ikx} dk = \mathcal{F}^{-1}\{F(k)\}$$

Displacements & Phase Shifts

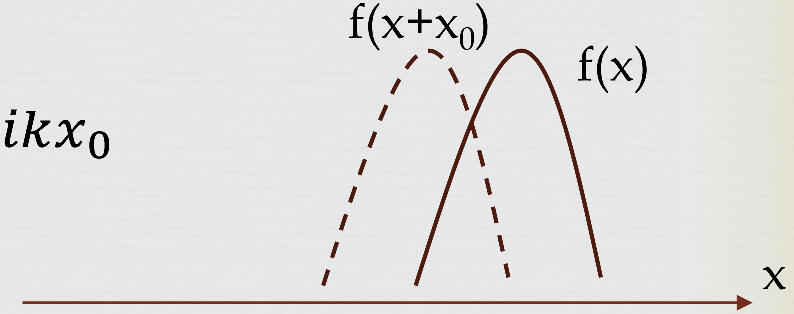
$$\mathcal{F}\{f(x+x_0)\} = \int_{-\infty}^{+\infty} f(x+x_0) e^{-ikx} dx$$

$x + x_0$=u

$$= \int_{-\infty}^{+\infty} f(u) e^{-ik(u-x_0)} du = e^{ikx_0} \int_{-\infty}^{+\infty} f(u) e^{-iku} du = e^{ikx_0} \mathcal{F}\{f(x)\}$$

$$\boxed{\mathcal{F}\{f(x+x_0)\} = e^{ikx_0} \mathcal{F}\{f(x)\}}$$

$$\mathcal{F}\{\delta(x+x_0)\} = e^{ikx_0} \mathcal{F}\{\delta(x)\} = e^{ikx_0}$$



$$F(k) = \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx = \mathcal{F}\{f(x)\}$$

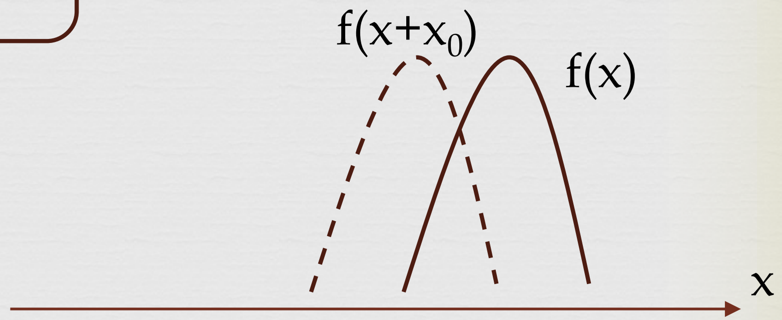
$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(k) e^{ikx} dk = \mathcal{F}^{-1}\{F(k)\}$$

Phase Shift & Displacements

$$\mathcal{F}\left\{f(x)e^{ik_0x}\right\} = \int_{-\infty}^{+\infty} f(x)e^{ik_0x}e^{-ikx}dx = \int_{-\infty}^{+\infty} f(x)e^{-i(k-k_0)x}dx$$

$$= F(k-k_0)$$

$$\boxed{\mathcal{F}\left\{f(x)e^{ik_0x}\right\} = F(k-k_0)}$$



$$\mathcal{F}\{1\} = \int_{-\infty}^{+\infty} e^{-ikx} dx = 2\pi\delta(k)$$

$$\mathcal{F}\left\{e^{ik_0x}\right\} = 2\pi\delta(k-k_0)$$

Sines and Cosines

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}\,dk = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ikx}\,dk$$

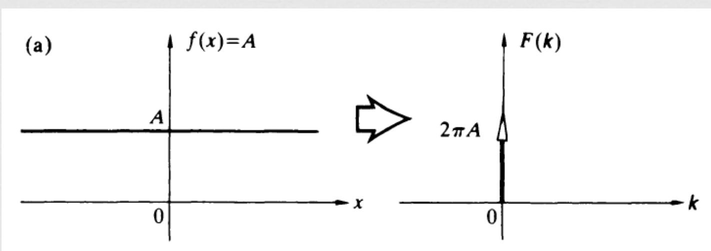
$$\mathcal{F}\{\delta(x)\} = 1$$

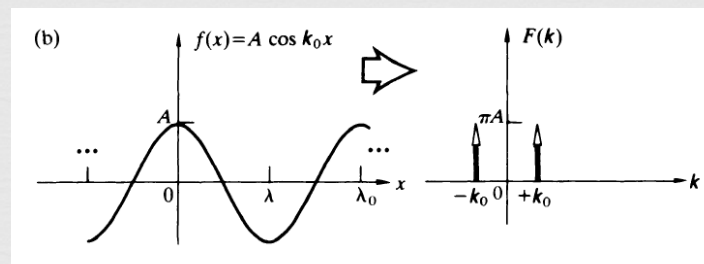
$$\mathcal{F}\{1\} = \int_{-\infty}^{+\infty} e^{-ikx}\,dx = 2\pi\delta(k)$$

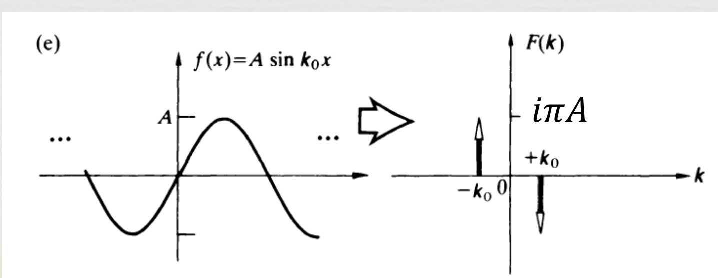
$$\mathcal{F}\{e^{ik_0x}\} = 2\pi\delta(k-k_0)$$

$$\sin k_0x = \frac{1}{2i}\left(e^{ik_0x} - e^{-ik_0x}\right) \longrightarrow \mathcal{F}\{\sin k_0x\} = -i\pi\delta(k-k_0) + i\pi\delta(k+k_0)$$

$$\cos k_0x = \frac{1}{2}\left(e^{ik_0x} + e^{-ik_0x}\right) \longrightarrow \mathcal{F}\{\cos k_0x\} = \pi\delta(k-k_0) + \pi\delta(k+k_0)$$







Sines and Cosines

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}dk = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ikx}dk$$

$$\mathcal{F}\{\delta(x)\} = 1$$

$$\mathcal{F}\{\delta(x+x_0)\} = e^{ikx_0}\mathcal{F}\{\delta(x)\}=e^{ikx_0}$$

$$\mathcal{F}\{\delta(x+x_0)+\delta(x-x_0)\} = e^{ikx_0}+e^{-ikx_0} = 2\cos(kx_0)$$

$$\mathcal{F}\{\delta(x+x_0)-\delta(x-x_0)\} = e^{ikx_0}-e^{-ikx_0} = 2i\sin(kx_0)$$

11.2.2 2D Fourier Transform

$$f(x,y) = \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{+\infty} F(k_x,k_y) e^{i(k_x x + k_y y)} dk_x\, dk_y$$

$$= \mathcal{F}^{-1}\left\{F(k_x,k_y)\right\}$$

$$F(k_x,k_y) = \iint_{-\infty}^{+\infty} f(x,y) e^{-i(k_x x + k_y y)} dx\, dy$$

$$= \mathcal{F}\{f(x,y)\}$$

FT of the Cylinder Function

$$f(x,y) = \begin{cases} 1 & \sqrt{x^2+y^2} \leq a \\ 0 & \sqrt{x^2+y^2} > a \end{cases}$$

$$k_x = k_\alpha \cos \alpha$$

$$k_y = k_\alpha \sin \alpha$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$F(k_\alpha) = 2\pi a^2 \left[\frac{J_1(k_\alpha a)}{k_\alpha a} \right]$$

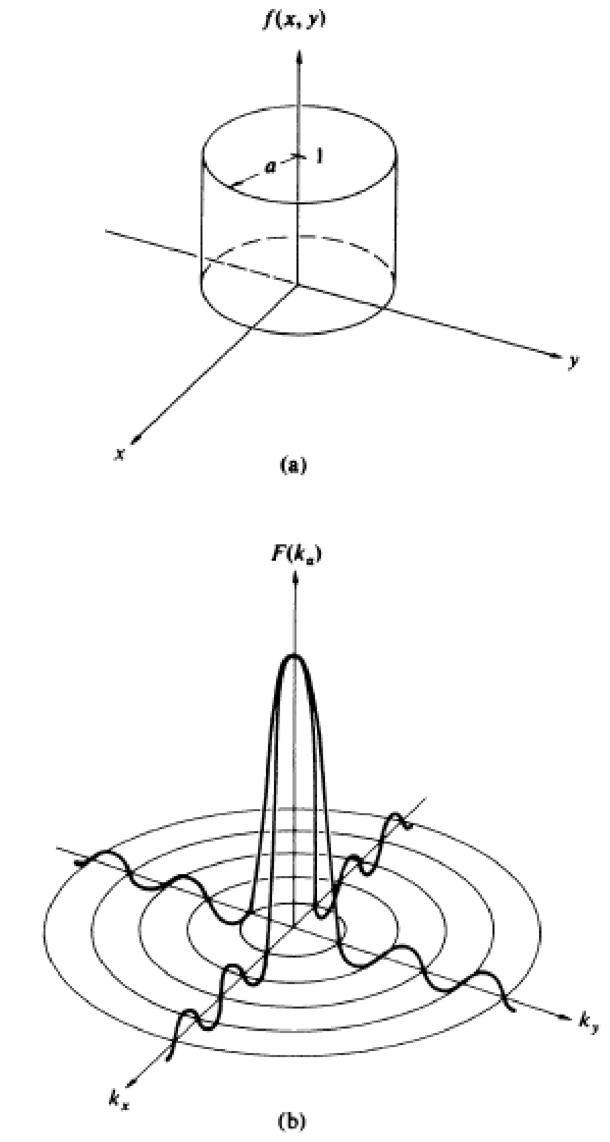
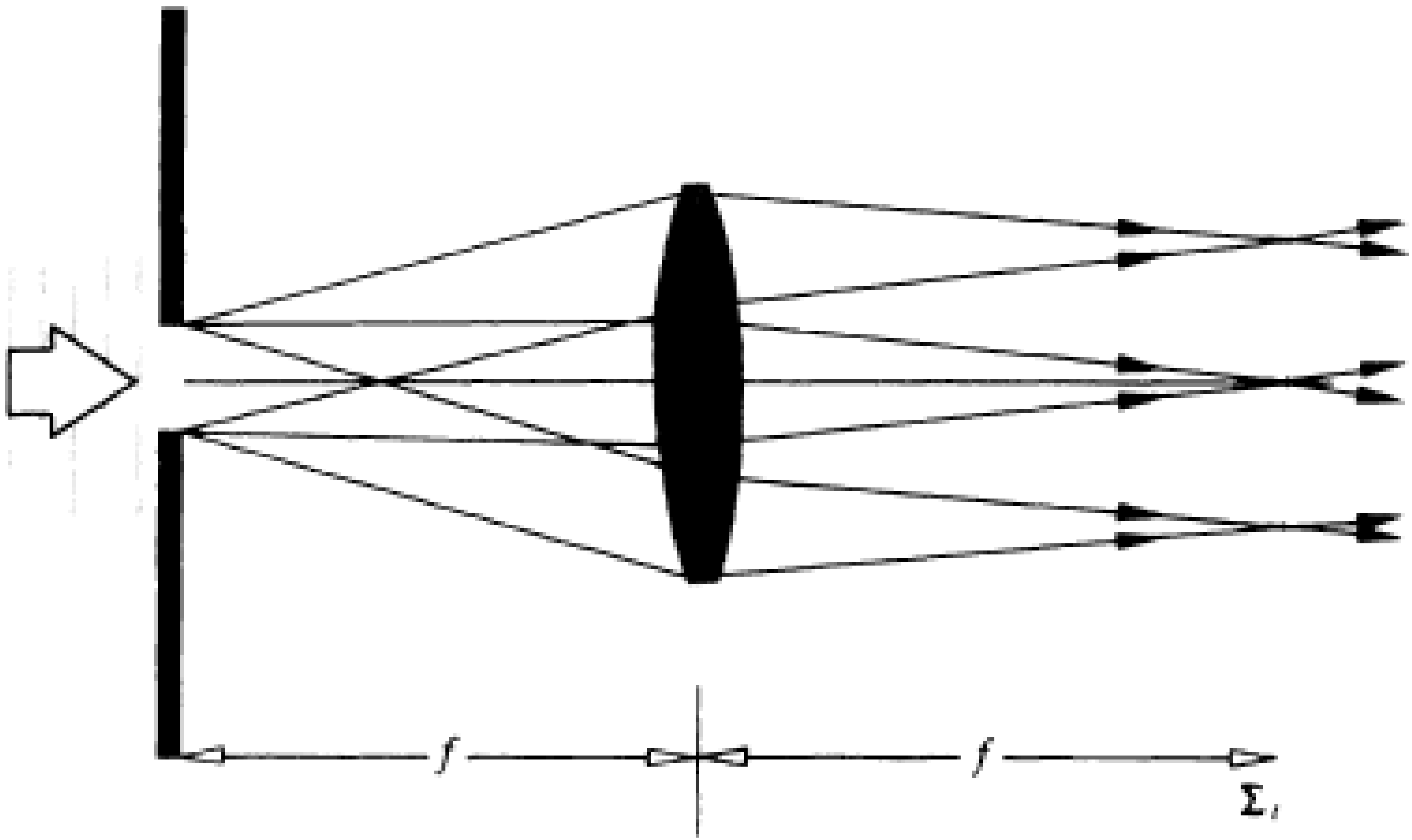


Figure 11.4 The cylinder, or top-hat, function and its transform.

The Lens as a FT



Optical Applications of FT

- Example: Free space –superposition principle

$$E(x,y,z,t)=f(x,y,z)e^{-i\omega t} \qquad k=\frac{\omega}{c}=\sqrt{k_x^2+k_y^2+k_z^2} \qquad k_z=\sqrt{k^2-k_x^2-k_y^2}$$

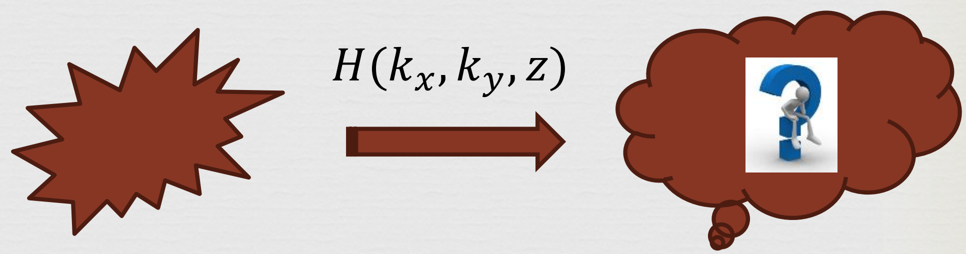
$$H(k_x,k_y,z)=e^{ik_z z}=e^{i\sqrt{k^2-k_x^2-k_y^2}z}$$

$$f(x,y,z)=\iint F(k_x,k_y)\,H(k_x,k_y,z)e^{i[k_x x+k_y y]}dk_x dk_y$$

$$f_0(x,y)=f(x,y,z=0)=\iint F(k_x,k_y)\,e^{i[k_x x+k_y y]}dk_x dk_y$$

$$F(k_x,k_y)=\mathcal{F}\{f_0(x,y)\}$$

$$F(k_x,k_y)H(k_x,k_y,z)=\mathcal{F}\{f(x,y,z)\}$$

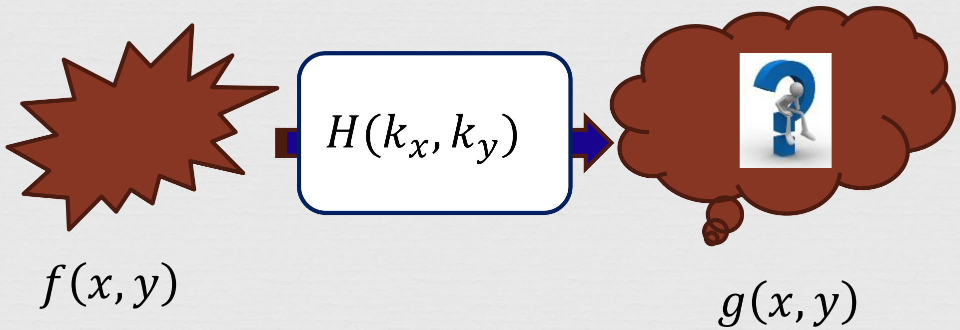


$f_0(x,y)$ $\qquad$ $f(x,y,z)$

$$\mathcal{F}^{-1}\{\mathcal{F}\{f_0(x,y)\}H(k_x,k_y,z)\}$$

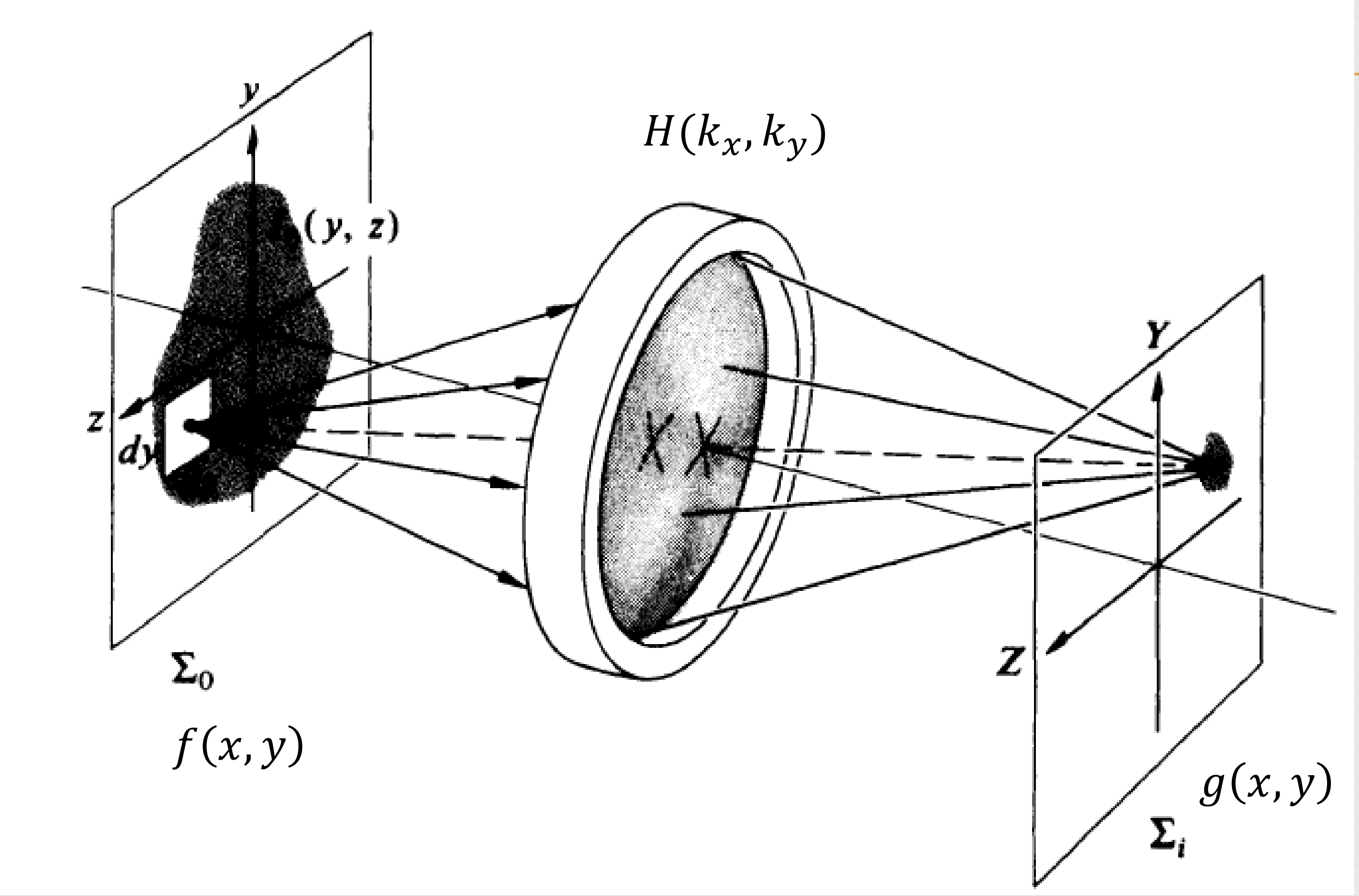
General Linear Systems

- Transform Function $H\left(k_x, k_y\right)$



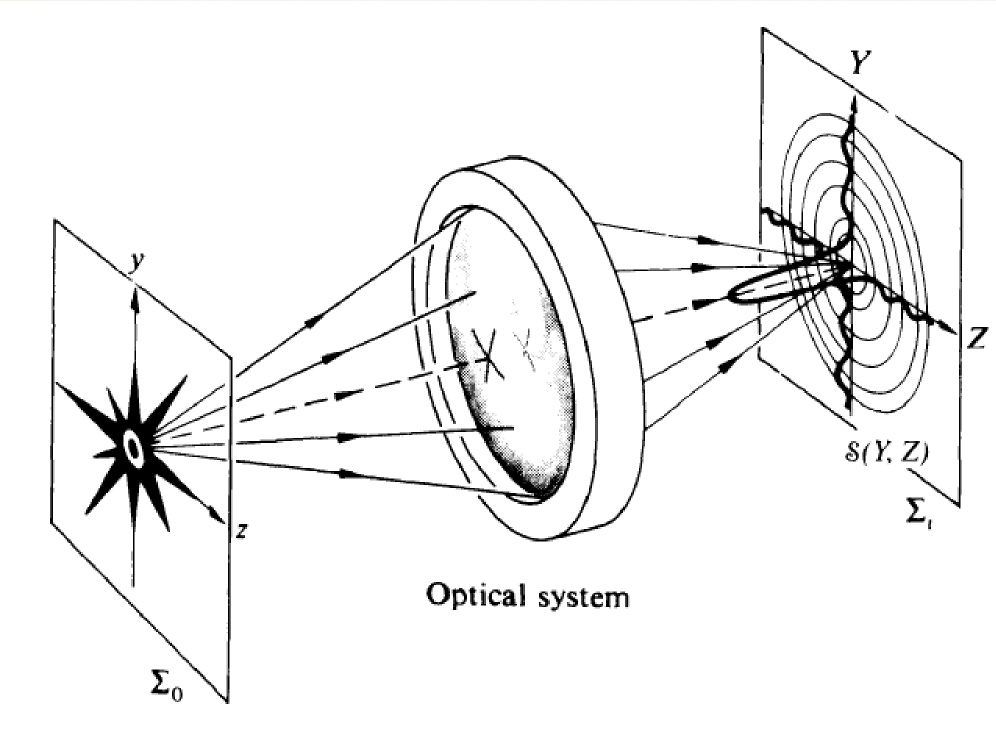
$$g(x, y) = \mathcal{F}^{-1}\{\mathcal{F}\{f(x, y)\}H(k_x, k_y)\}$$

A Lens Imaging System



$$g(x,y) = \mathcal{F}^{-1}\{\mathcal{F}\{f(x,y)\}H(k_x,k_y)\}$$

Point Spread Function



$$PSF(x,y) = \mathcal{F}^{-1}\{H(k_x,k_y)\}$$

➔ Resolution

$$PSF(x,y) = \mathcal{F}^{-1}\{\mathcal{F}\{\delta(x,y)\}H(k_x,k_y)\}$$
$$= \mathcal{F}^{-1}\{H(k_x,k_y)\}$$