PHYS 3038 Optics
L18 Diffraction
Reading Material: Ch10.2-2

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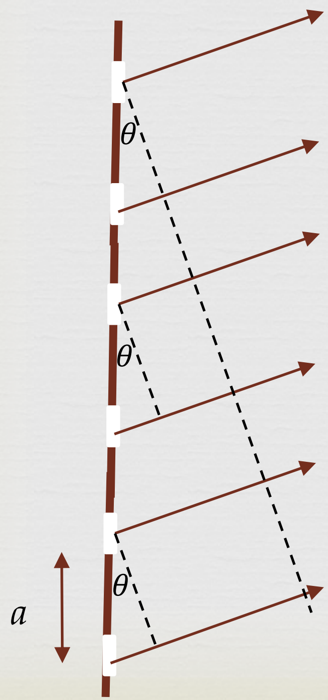
2015, the Year of Light

10.2.3 Fraunhofer Diffraction by many Slits

Recall: single slit $$E_1 = \frac{D\mathcal{E}_L}{R} e^{ikR} e^{-i\omega t} \operatorname{sinc}\beta$$

$$\delta = ka\sin\theta \qquad \alpha = \delta/2$$



Diffraction + Interference

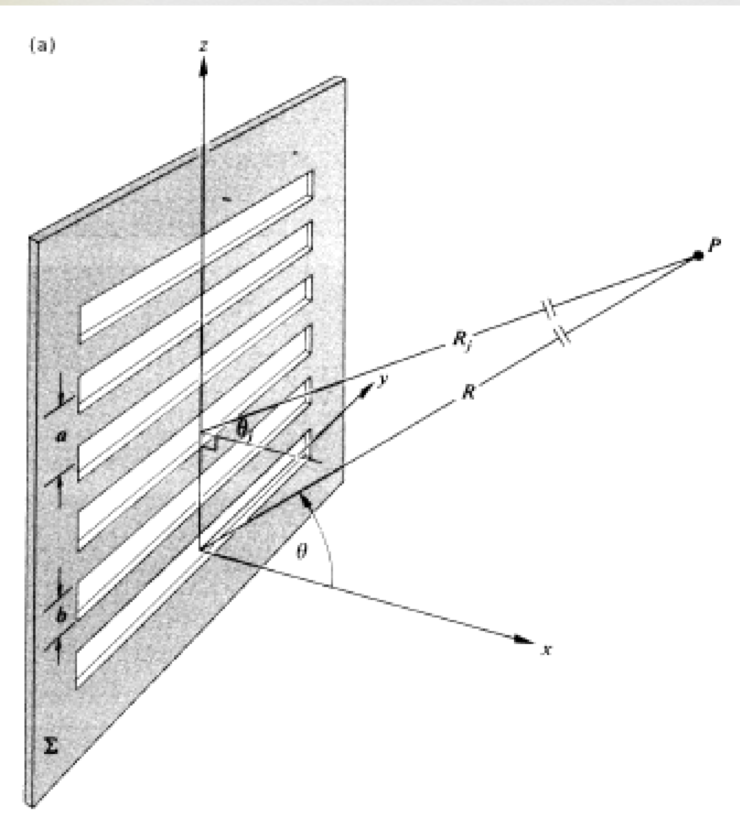
$$E = \sum_{m=1}^{N} E_m = \sum_{m=1}^{N} E_1 e^{i(m-1)\delta} = E_1 \sum_{m=0}^{N-1} e^{im\delta} = E_1 \frac{1-e^{iN\delta}}{1-e^{i\delta}}$$

$$= E_1 \frac{e^{iN\delta/2}}{e^{i\delta/2}} \frac{\sin N\delta/2}{\sin\delta/2} = E_1 \frac{\sin N\alpha}{\sin\alpha} e^{i(N-1)\delta/2}$$

$$I = \frac{1}{2}E^*E = \frac{1}{2}E_1^*E_1\left(\frac{\sin N\alpha}{\sin\alpha}\right)^2 = I_1\left(\frac{\sin N\alpha}{\sin\alpha}\right)^2 = I_0\left(\frac{\sin\beta}{\beta}\right)^2\left(\frac{\sin N\alpha}{\sin\alpha}\right)^2$$

Many Slits

$$I = I_0 \left(\frac{\sin \beta}{\beta}\right)^2 \left(\frac{\sin N\alpha}{\sin \alpha}\right)^2$$



lier treatment [Eq. (10.17)], **principal maxima** occur when $(\sin N\alpha/\sin \alpha) = N$, that is, when

$$\alpha = 0, \pm\pi, \pm 2\pi, \ldots$$

or equivalently, since $\alpha = (ka/2) \sin \theta$,

$$a \sin \theta_m = m\lambda \qquad (10.32)$$

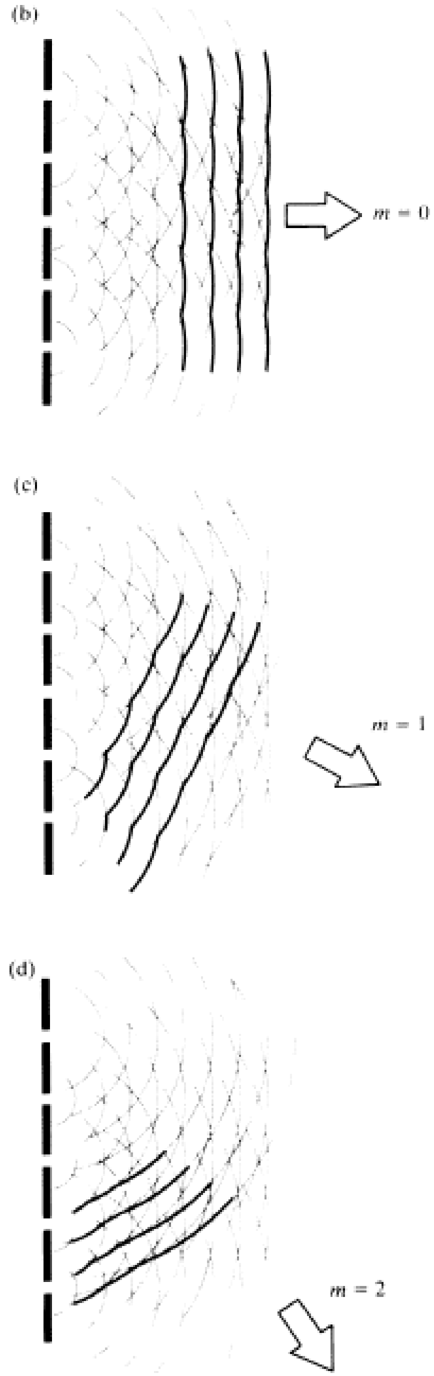
with $m = 0, \pm 1, \pm 2, \ldots$. This is quite general and gives rise to the same θ-locations for these maxima, regardless of the value of $N \geq 2$. Minima, of zero flux density, exist whenever $(\sin N\alpha/\sin \alpha)^2 = 0$ or when

$$\alpha = \pm\frac{\pi}{N}, \pm\frac{2\pi}{N}, \pm\frac{3\pi}{N}, \ldots, \pm\frac{(N-1)\pi}{N}, \pm\frac{(N+1)\pi}{N}, \ldots \qquad (10.33)$$

Between consecutive principal maxima (i.e., over the range in α of π) there will therefore be $N - 1$ minima. And, of course, between each pair of minima there will have to be a **subsidiary maximum**. The term $(\sin N\alpha/\sin \alpha)^2$, which we can think of as embodying the interference effects, has a rapidly varying numerator and a slowly varying denominator. The subsidiary maxima are therefore located approximately at points where $\sin N\alpha$ has its greatest value, namely,

$$\alpha = \pm\frac{3\pi}{2N}, \pm\frac{5\pi}{2N}, \ldots \qquad (10.34)$$

The $N - 2$ *subsidiary maxima* between consecutive principal maxima are clearly visible in Fig. 10.16. We can get some idea



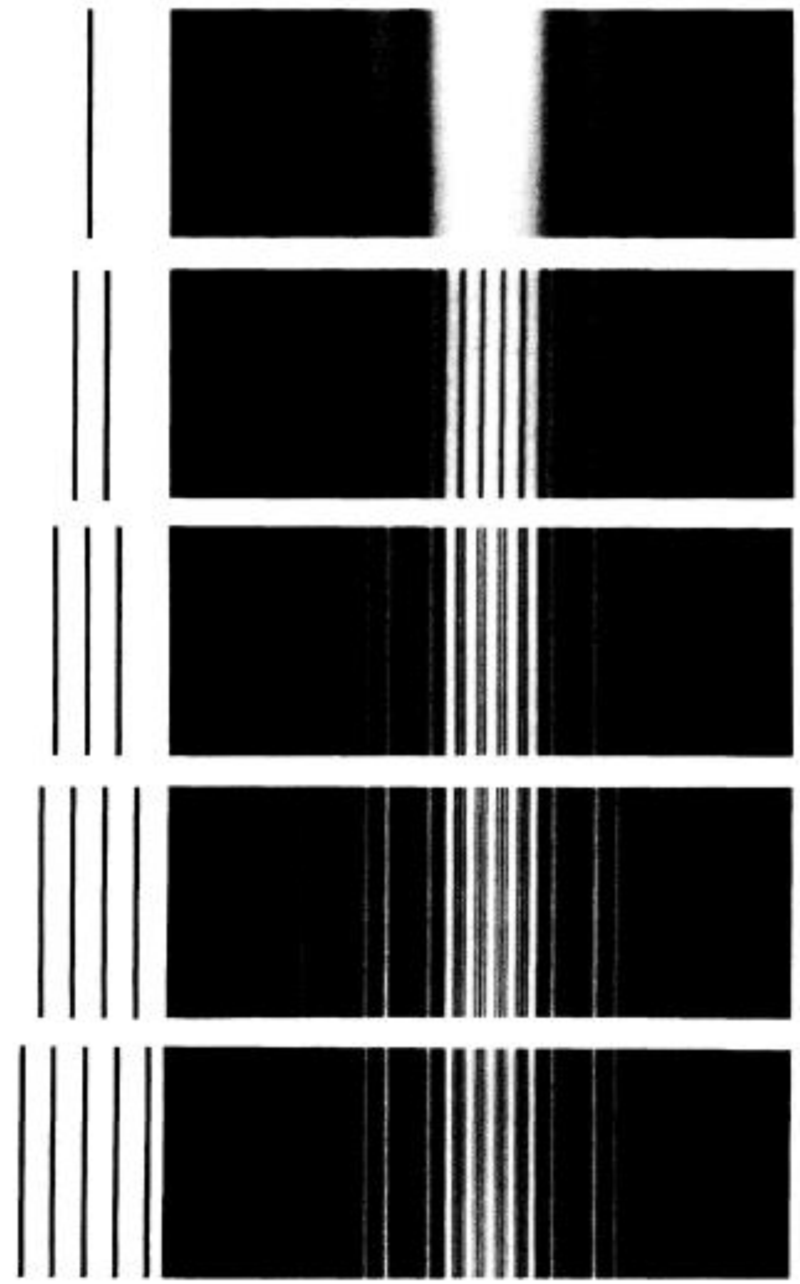
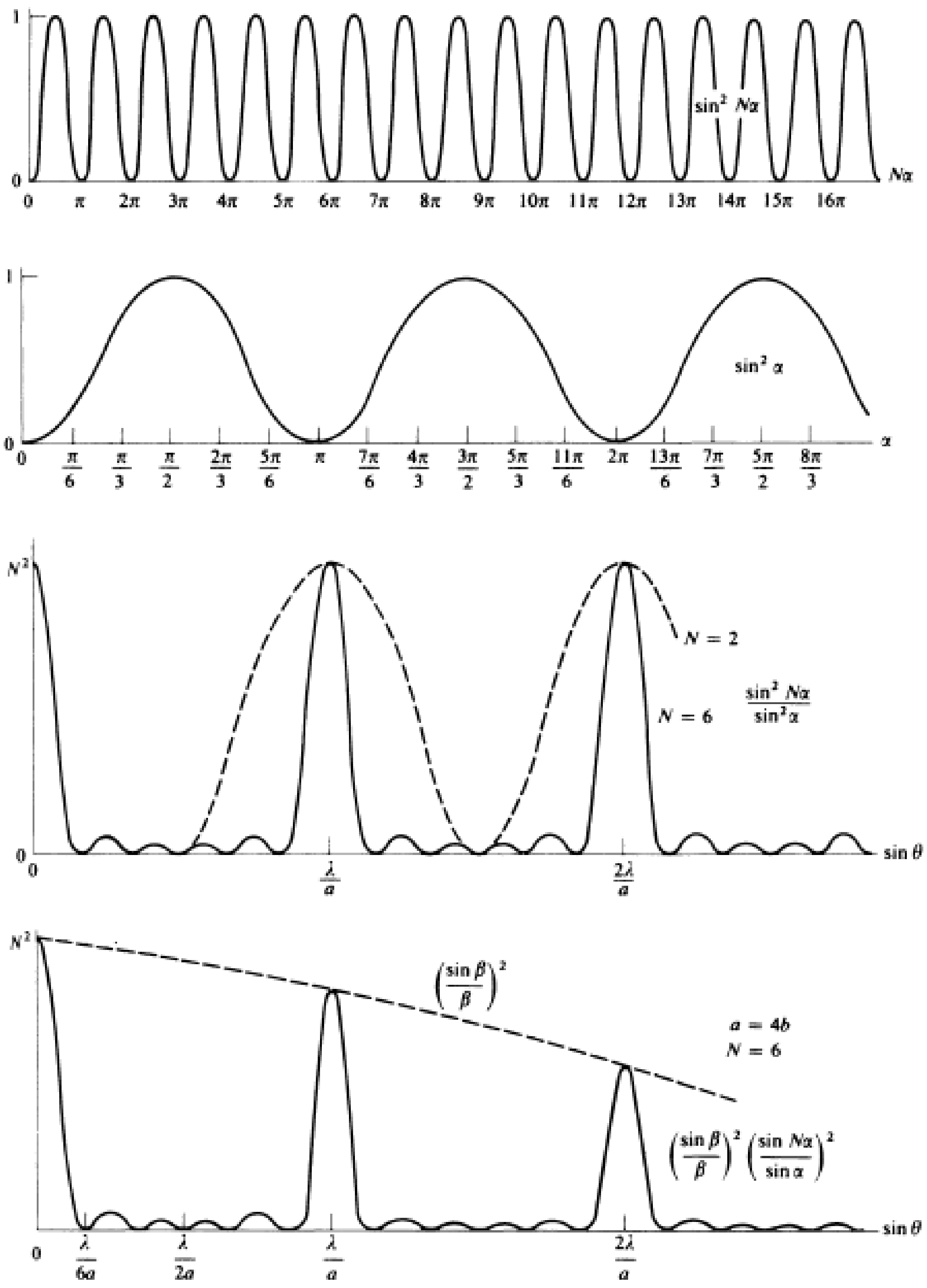


Figure 10.16 Diffraction patterns for slit systems shown at left. (Francis Weston Sears, *Optics*. Reprinted with permission of Addison Wesley Longman, Inc.)



$$I = I_0 \left(\frac{\sin\beta}{\beta}\right)^2 \left(\frac{\sin N\alpha}{\sin\alpha}\right)^2$$

10.2.4 2D Aperture

$$E = \iint \frac{\mathcal{E}_A}{r} e^{i(kr-\omega t)} dS$$

$$\cong \frac{\mathcal{E}_A}{R} e^{i(kR-\omega t)} \iint e^{-ik(Xx+Yy)/R} dS$$

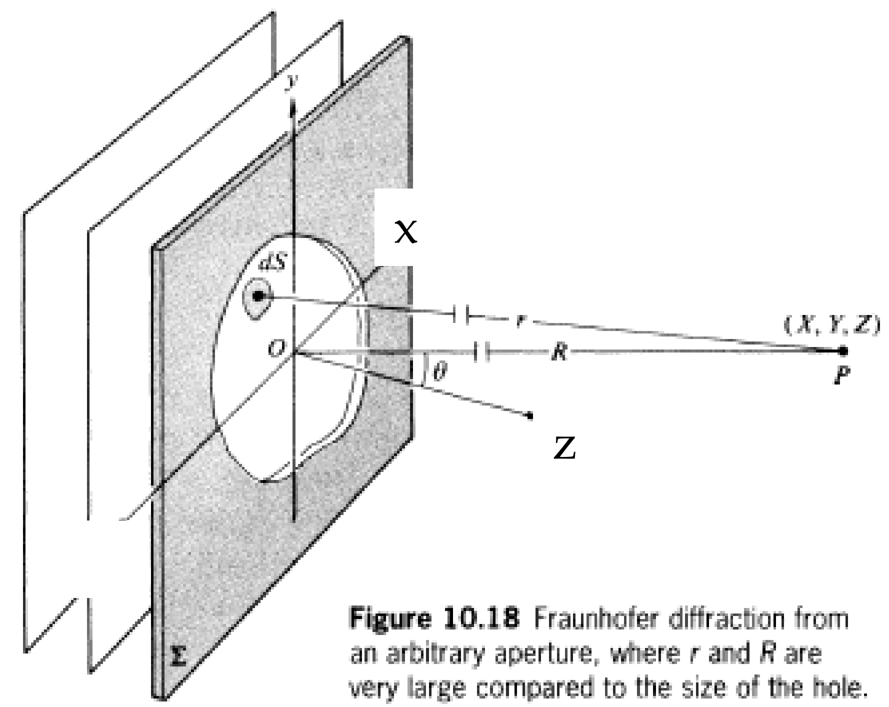


Figure 10.18 Fraunhofer diffraction from an arbitrary aperture, where *r* and *R* are very large compared to the size of the hole.

Rectangular Aperture

$$E_1 = \frac{D\mathcal{E}_L}{R} e^{ikR} e^{-i\omega t} \operatorname{sinc}\beta$$

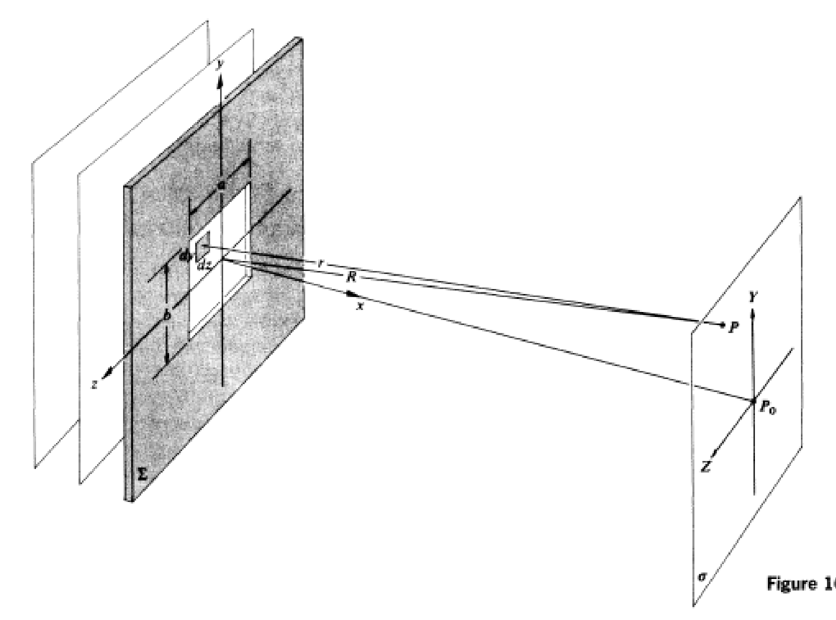


Figure 10

$$E \cong \frac{\mathcal{E}_A}{R} e^{i(kR-\omega t)} \iint e^{-ik(Xx+Yy)/R} dS$$

$$= \frac{\mathcal{E}_L}{R} e^{i(kR-\omega t)} \int_{-a/2}^{a/2} e^{-ikXx/R} dx \int_{-b/2}^{b/2} e^{-ikYy/R} dy$$

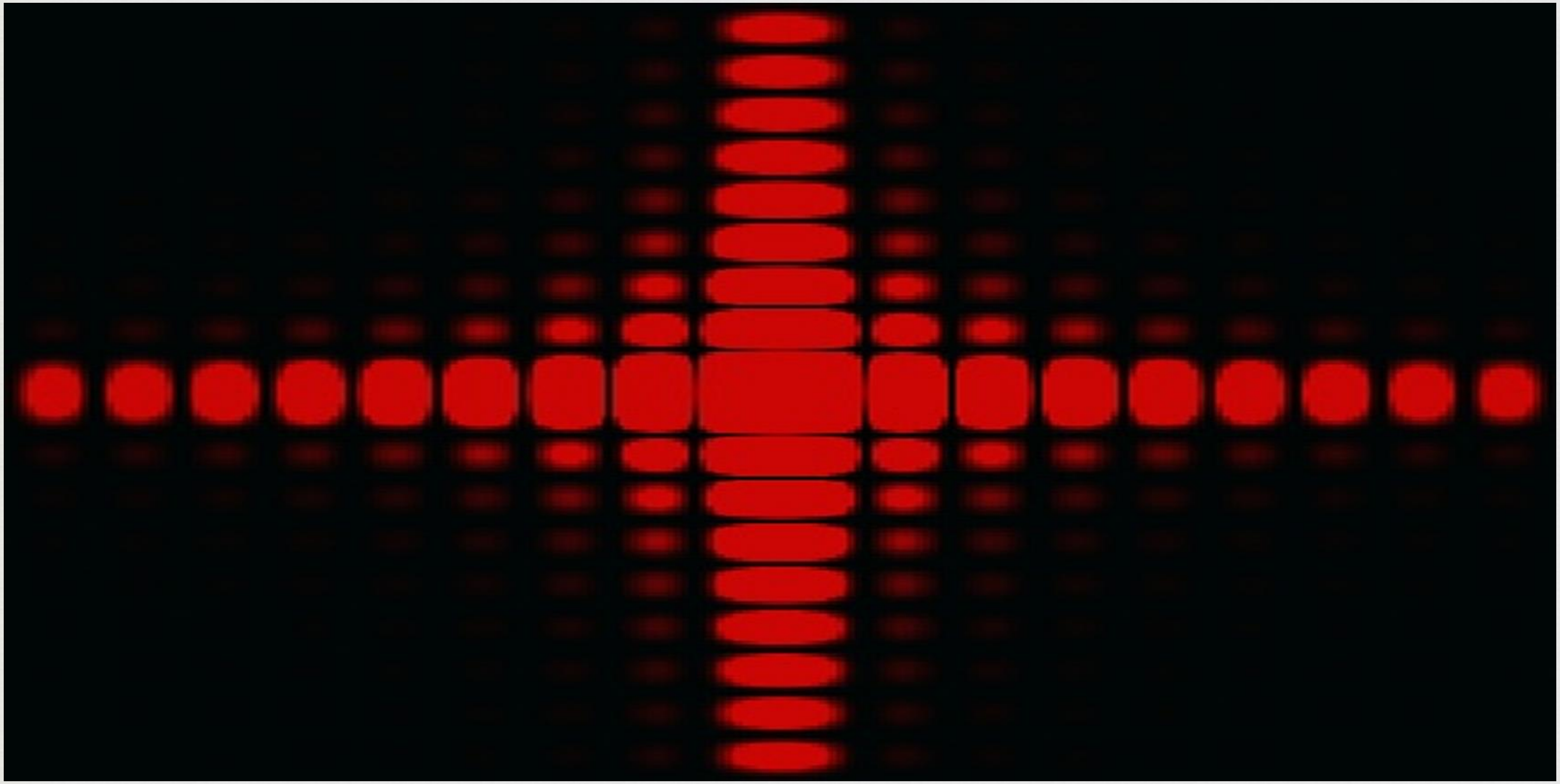
$$= \frac{\mathcal{E}_L}{R} e^{i(kR-\omega t)} \int_{-a/2}^{a/2} e^{-ikXx/R} dx \int_{-b/2}^{b/2} e^{-ikYy/R} dy$$

$$= \frac{ab\mathcal{E}_A}{R} e^{ikR} e^{-i\omega t} \operatorname{sinc}\alpha \operatorname{sinc}\beta$$

$$I = I_0 \operatorname{sinc}^2\alpha \operatorname{sinc}^2\beta$$

$$\alpha = \left(\frac{ka}{2}\right) \sin\theta_x = \frac{kaX}{2R} \qquad \beta = \left(\frac{kb}{2}\right) \sin\theta_x = \frac{kbY}{2R}$$

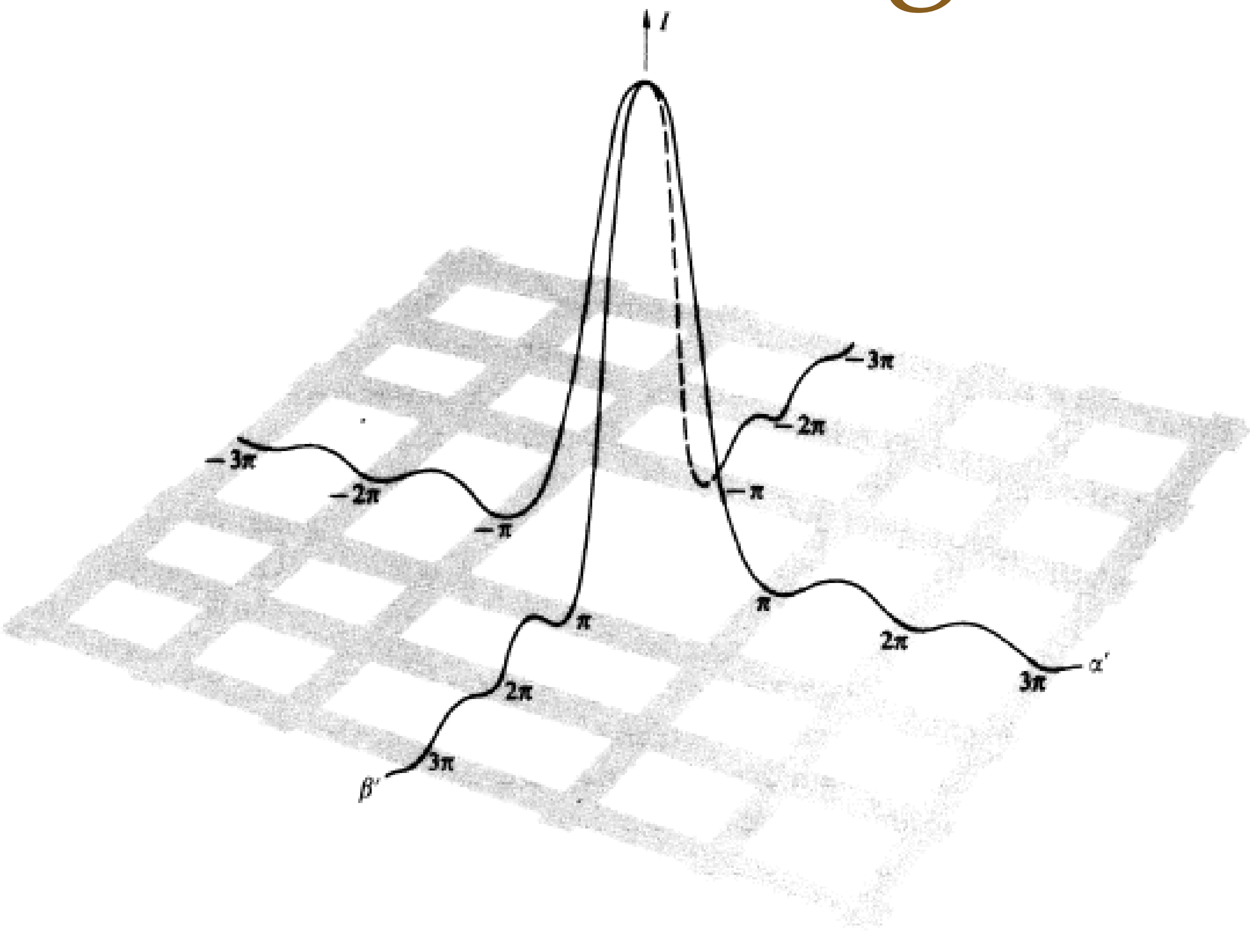
Rectangular Slit



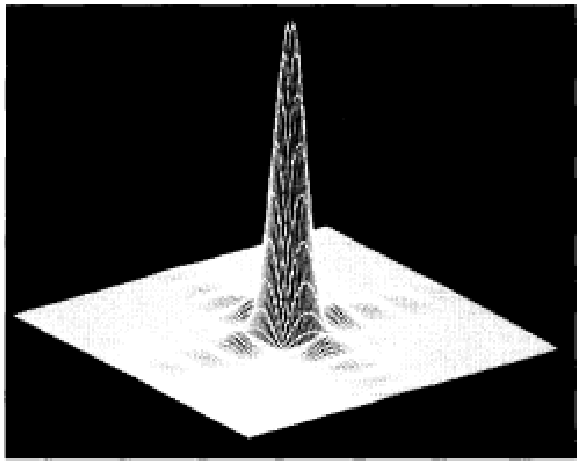
$I = I_0 \operatorname{sinc}^2 \beta_x \operatorname{sinc}^2 \beta_y$

$\beta_{x,y} = (\frac{kD_{x,y}}{2}) \sin \theta_{x,y}$

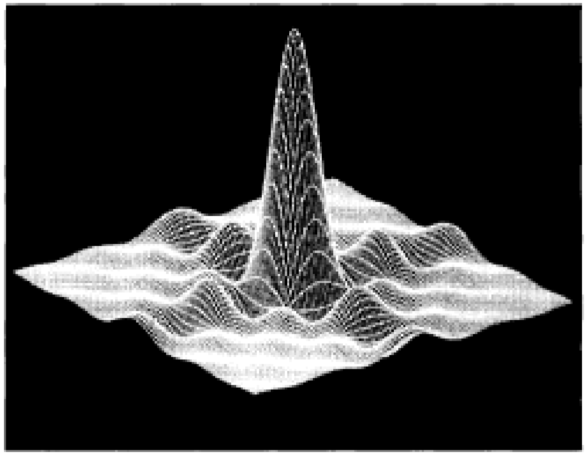
Rectangular Slit



(a)



(b)



(c)

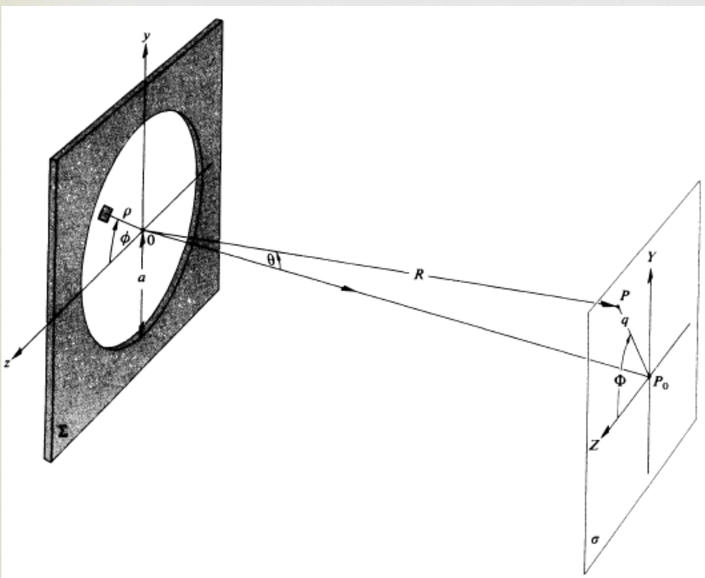
Figure 10.20 (*a*) The irradiance distribution for a square aperture. (*b*) The irradiance produced by Fraunhofer diffraction at a square aperture. (*c*) The electric-field distribution produced by Fraunhofer diffraction via a square aperture. (Photos courtesy R. G. Wilson, Illinois Wesleyan University.)

Circular Aperture

$$E \cong \frac{\mathcal{E}_A}{R} e^{i(kR-\omega t)} \iint e^{-ik(Xx+Yy)/R} dS$$

$x = \rho \cos \phi$ $\quad$ $y = \rho \sin \phi$

$X = q \cos \Phi$ $\quad$ $Y = q \sin \Phi$



$$E = \frac{\mathcal{E}_A}{R} e^{i(kR-\omega t)} \int_0^{2\pi} \int_0^{a} e^{-i\left(\frac{k\rho q}{R}\right)\cos(\phi - \Phi)} \rho d\rho d\phi$$

The Bessel function (of the first kind) of order zero

$$J_0(u) = \frac{1}{2\pi} \int_0^{2\pi} e^{iu\cos v} dv$$

$$E = \frac{\mathcal{E}_A}{R} e^{i(kR-\omega t)} 2\pi \int_0^{a} J_0\left(\frac{k\rho q}{R}\right) \rho d\rho$$

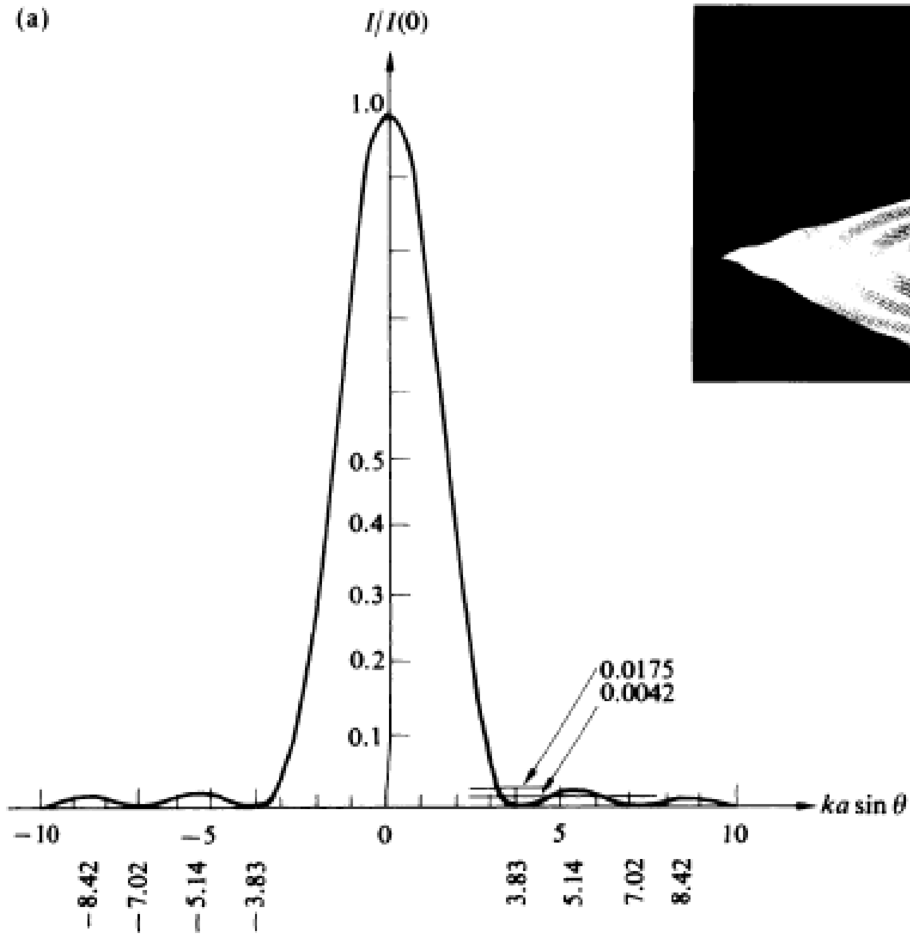
$$= \frac{\mathcal{E}_A}{R} e^{i(kR-\omega t)} 2\pi a^2 \frac{J_1\left(\frac{kaq}{R}\right)}{\frac{kaq}{R}}$$

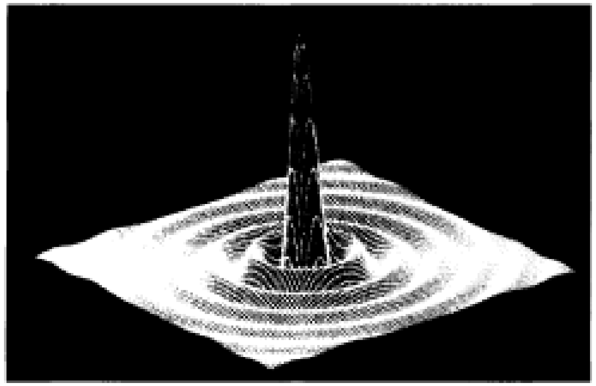
$$I = \frac{2(\mathcal{E}_A A)^2}{R^2} \left[\frac{J_1\left(\frac{kaq}{R}\right)}{\frac{kaq}{R}}\right]^2$$

$\sin\theta = q/R$ $\longrightarrow$ $I_0 \left[\frac{J_1(ka\sin\theta)}{ka\sin\theta}\right]^2$

$$I_0\left[\frac{J_1(ka\sin\theta\ )}{ka\sin\theta}\right]^2$$

Airy Disk





(b)

(c)

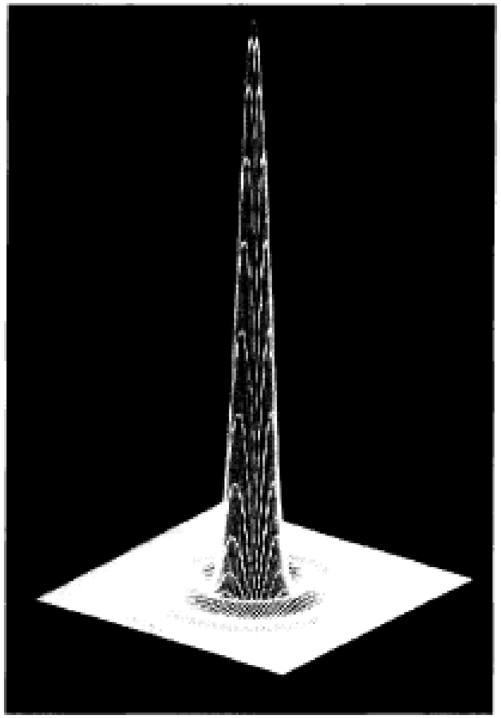
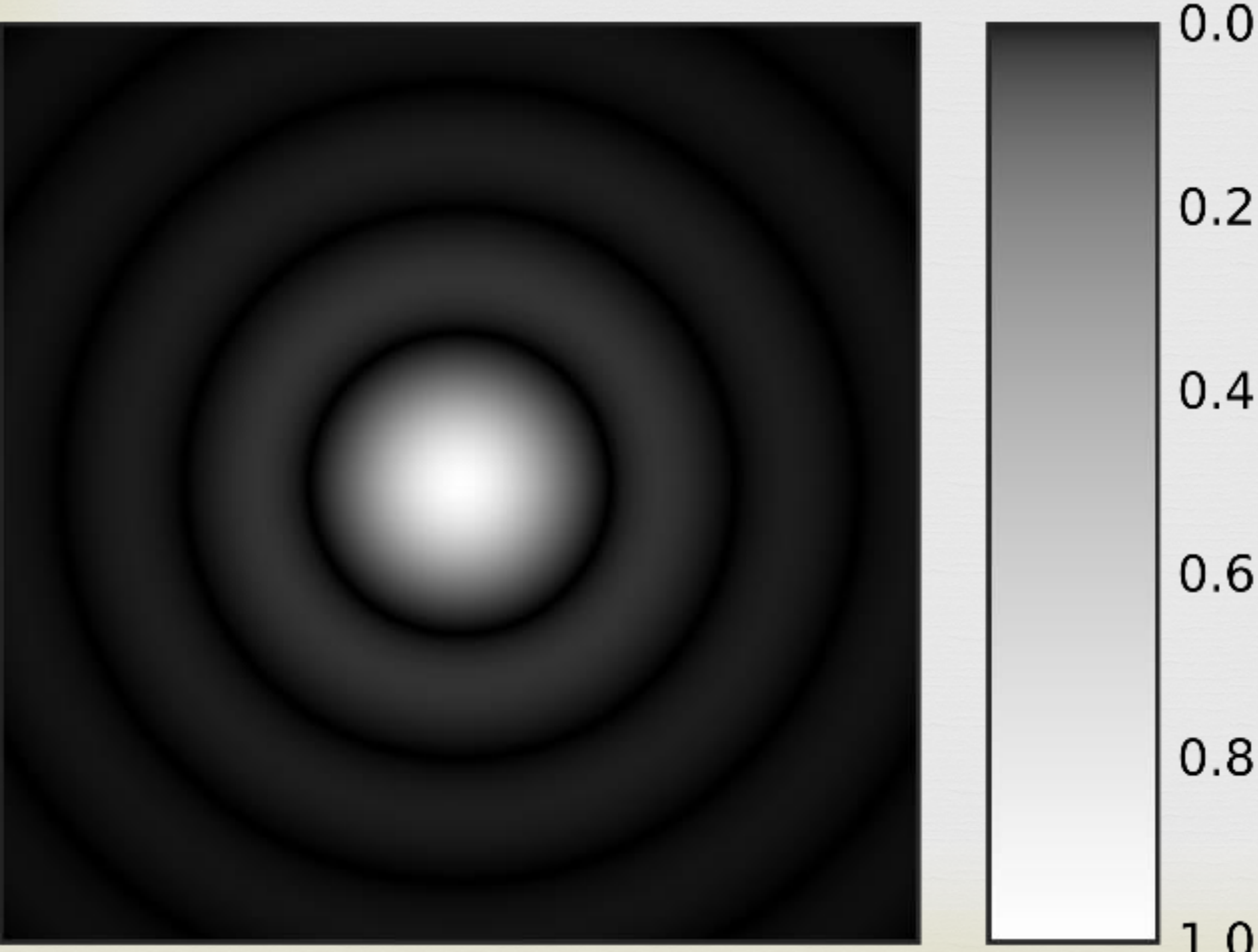
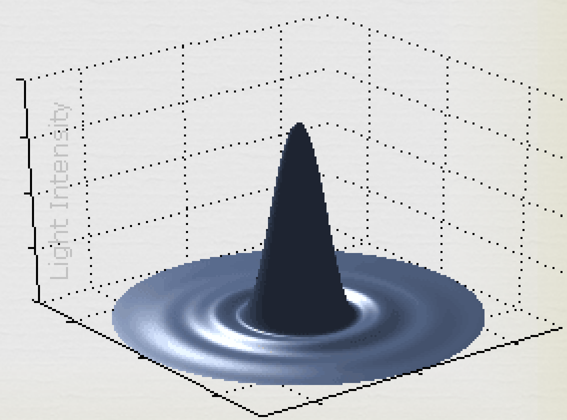


Figure 10.23 (*a*) The Airy pattern. (*b*) Electric field created by Fraunhofer diffraction at a circular aperture. (*c*) Irradiance resulting from Fraunhofer diffraction at a circular aperture. (Photos courtesy R. G. Wilson, Illinois Wesleyan University.)

Airy Disk

$$I_0\left[\frac{J_1(ka\sin\theta\ )}{ka\sin\theta}\right]^2$$





Optical Imaging, Diffraction Limit, & Resolution

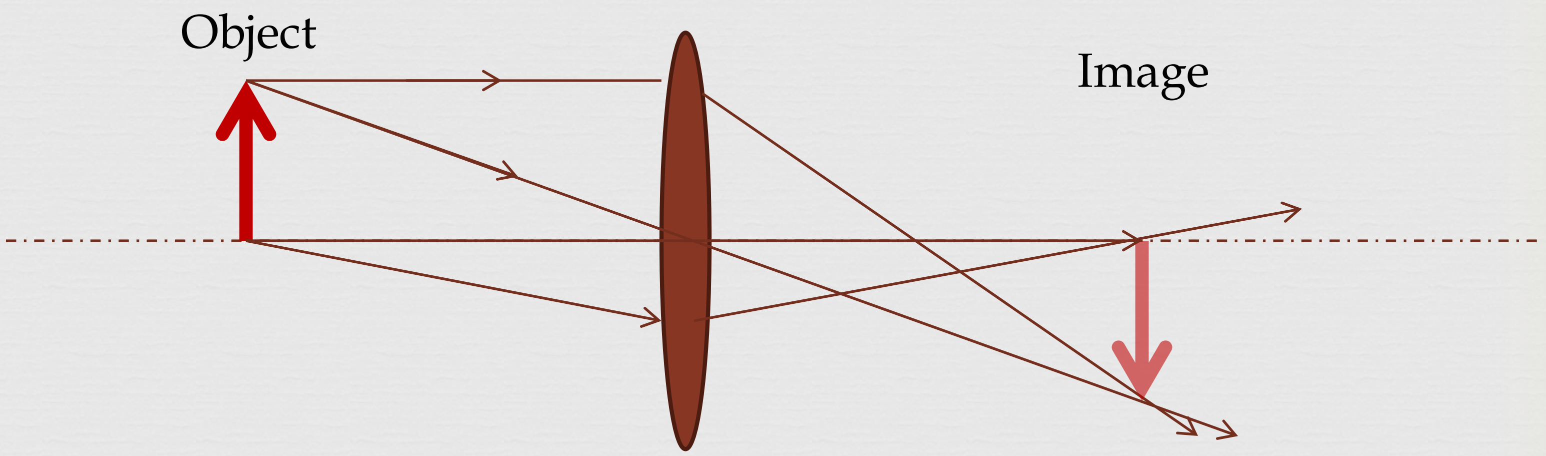
— *Wave Nature of Light*

Image Resolution

- Sensor resolution (pixel resolution)
- Spatial resolution

Optical Imaging

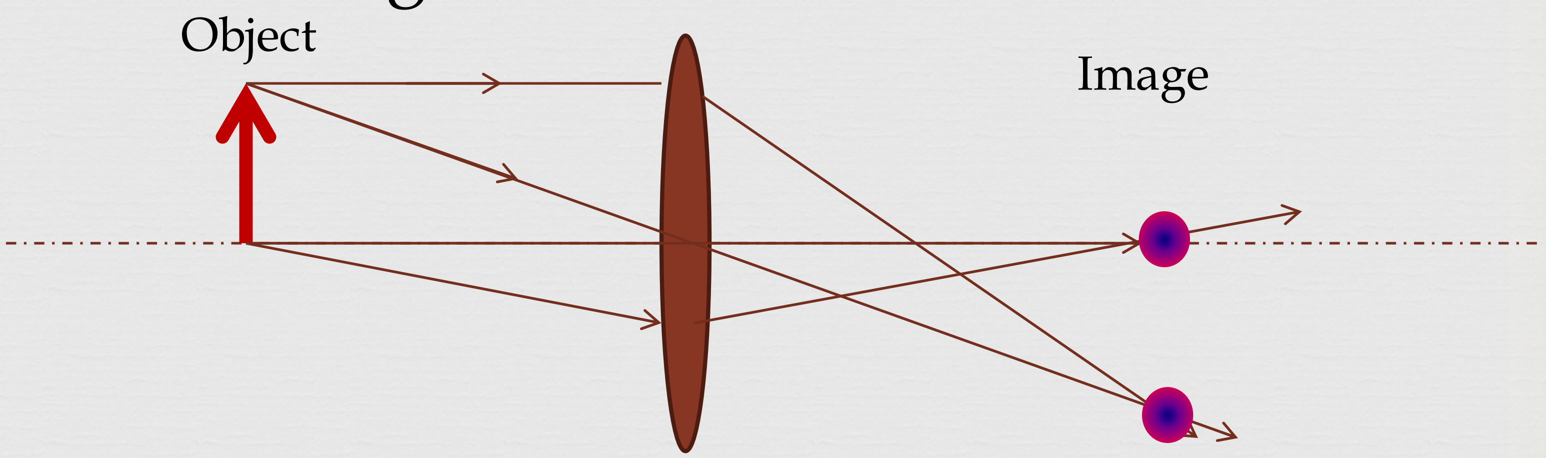
Optical Imaging Principle



Geometric Optics: point (object plane) ⇔ point (imaging plane)

Imaging Resolution: How small are we able to see?

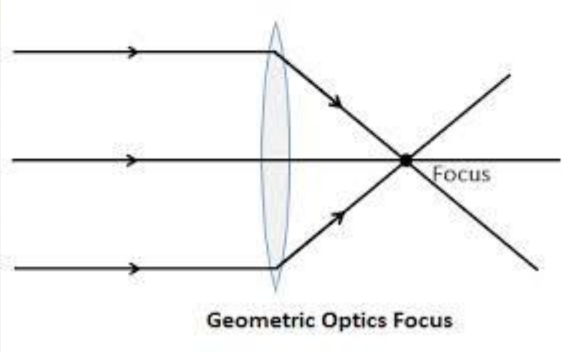
Remember: light is wave!



Wave Optics: point (object plane) ⇒ spot (imaging plane)

For illustration, we set image amplification as 1 to show the resolution.

How small can we make a light spot?



2D ideal sizeless point at (x, y):

$$\delta(x,y) = \delta(x)\delta(y) \qquad \delta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}dk$$

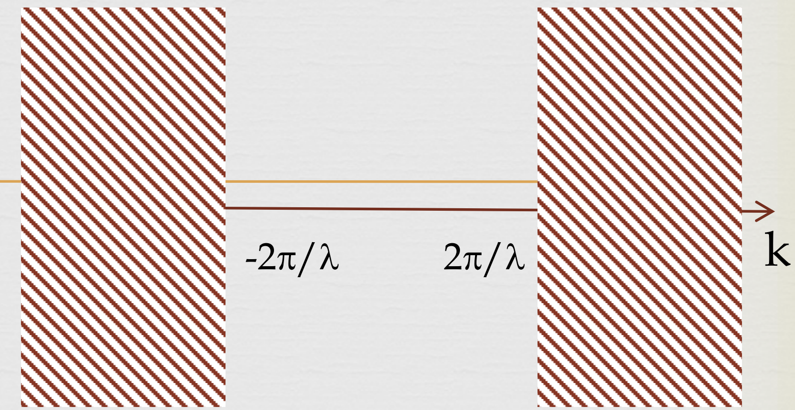
$$\delta(x,y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i(k_x x+k_y y)}dk_x\, d\,k_y$$

However…

$${k_x}^2 + {k_y}^2 \le {k_x}^2 + {k_y}^2 + {k_z}^2 = \left(\frac{\omega}{c}\right)^2 = \left(\frac{2\pi}{\lambda}\right)^2$$

1D

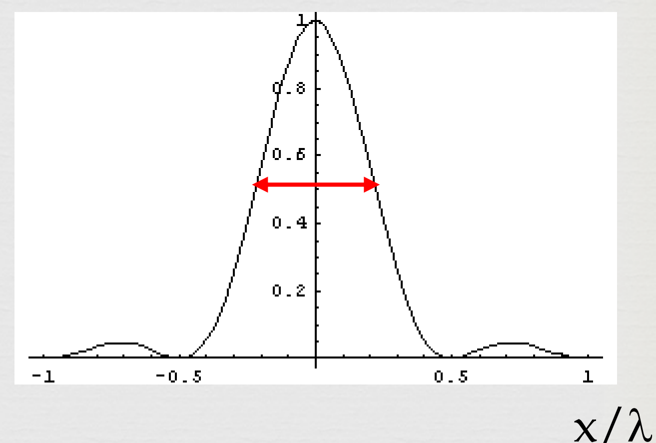
Let's start with 1D case for simple math



$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}\,dk$$

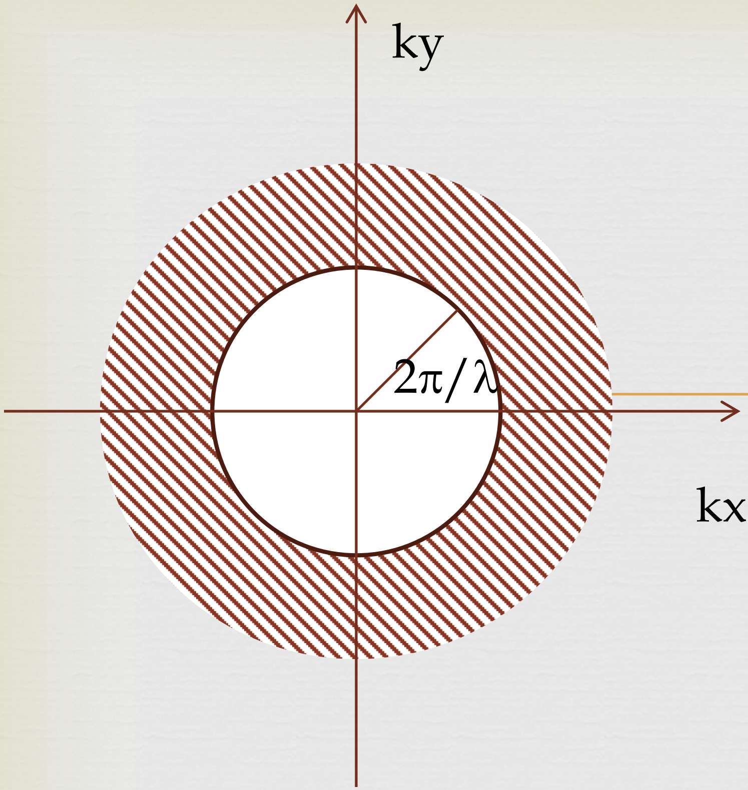
$$s(x) = \frac{1}{2\pi}\int_{-2\pi/\lambda}^{+2\pi/\lambda} e^{ikx}\,dk = \frac{\sin\left[\frac{2\pi x}{\lambda}\right]}{\pi x}$$

$$P(x) = |s(x)|^2 = \left|\frac{\sin\left[\frac{2\pi x}{\lambda}\right]}{\pi x}\right|^2$$



FWHM=0.44λ

2D

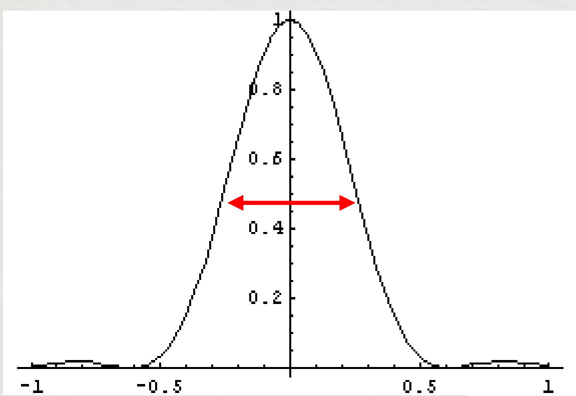


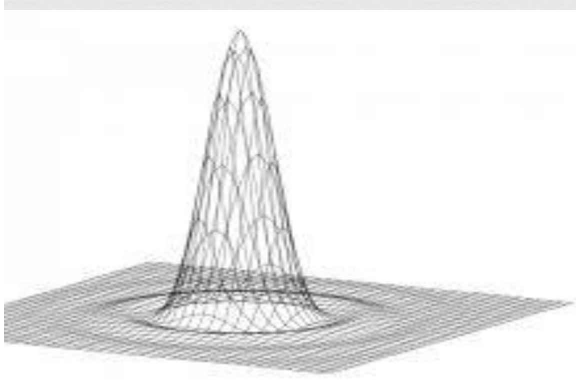
$$\rho = \sqrt{x^2 + y^2}$$

$J_1(2\pi\rho/\lambda)$: Bessel function of the first kind, order 1

$$s(\rho) = \frac{J_1(2\pi\rho/\lambda)}{\rho}$$

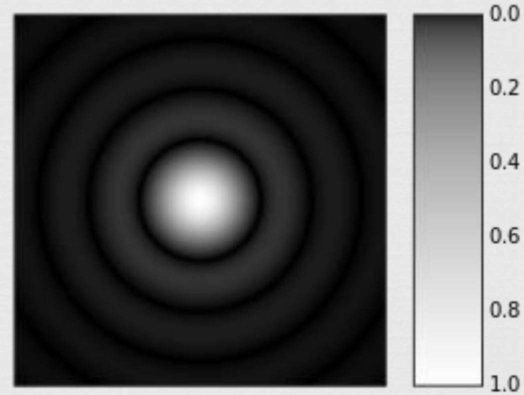
$$P(\rho) = |s(\rho)|^2 = \left|\frac{J_1(2\pi\rho/\lambda)}{\rho}\right|^2$$





ρ/λ

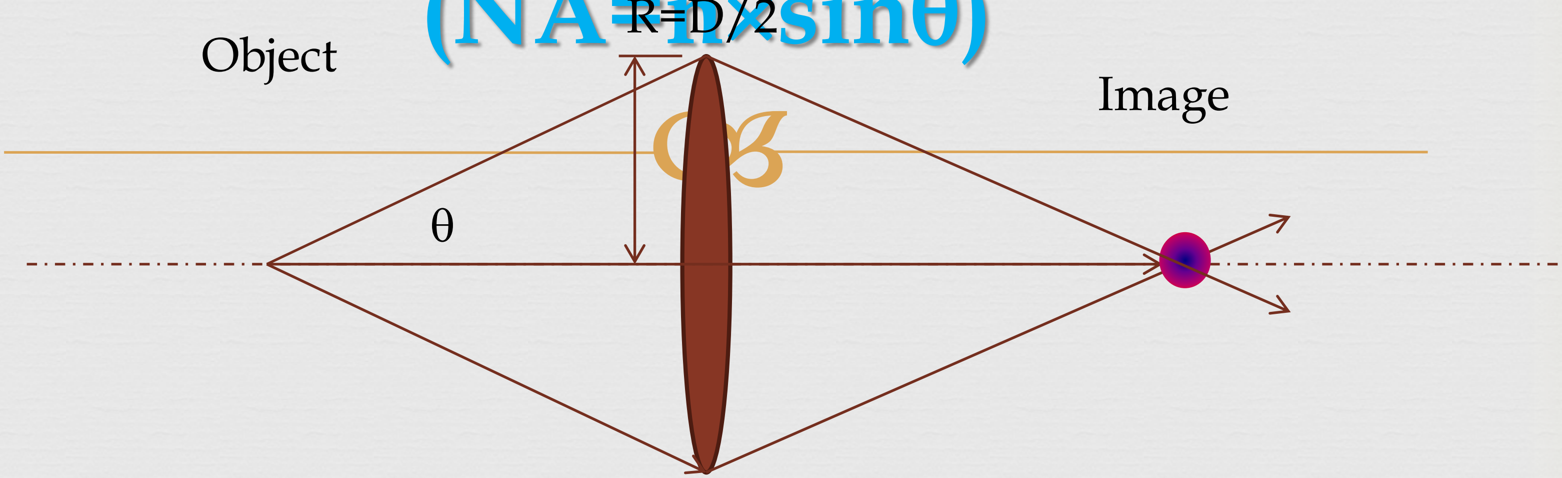
FWHM=0.5λ



Airy disk pattern

A Real Imaging System

with Numerical Aperature (NA=n×sinθ)



$$k_x^{\ 2} + k_y^{\ 2} \le \left(\frac{2\pi}{\lambda}\right)^2 \sin^2\theta = \left(\frac{2\pi}{\lambda}\sin\theta\right)^2$$

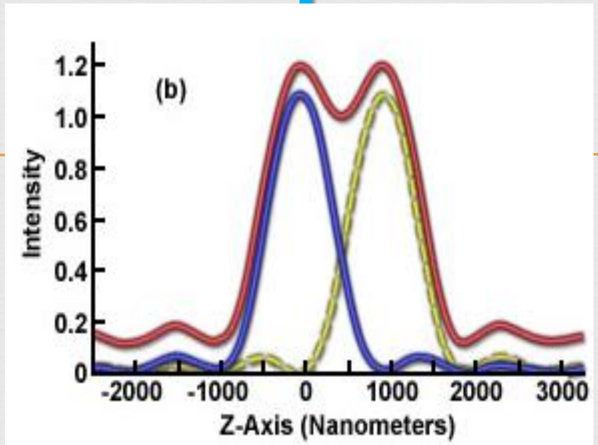
By replacing λ with λ/sinθ , we get

FWHM=0.5 λ/sinθ = λ/(2sinθ)

Further including the refractive index n of the imaging medium (λ→λ/n), we obtain

FWHM=Abbe Resolution= λ/(2nsinθ)=λ/(2NA)

A Real Imaging System
with Numerical Aperature (NA)



$NA = n\sin\theta \leq n$

$FWHM = \lambda/(2NA) \geq \lambda/(2n) \Rightarrow \lambda/2$ in air

Optical microscope (λ: 400 nm – 1000 nm):
best achievable resolution is 200 nm

Optical Microscope

Ernst Abbe is credited by many for discovering the resolution limit of the microscope, and the formula (published in 1873): 'Abbe limit'

Microscope by Carl Zeiss (1879) with optics by Abbe

$$d = \frac{\lambda}{2NA}$$

9

http://en.wikipedia.org/wiki/Ernst_Abbe

Nikon inverted microscope

Magnification vs. Resolution

In principle, one can achieve a magnification as large as possible by combining different lenses (then the FWHM of the image spot we discussed previously should also multiply the same magnification factor). There is a limit to the resolution no matter how big the magnification is. This is because light is wave.

Diffraction Limit

$d=\lambda/(2NA)$ 200 nm for visible light