PHYS 3038 Optics
L12 Polarization (continued)
Reading Material: Ch8.13

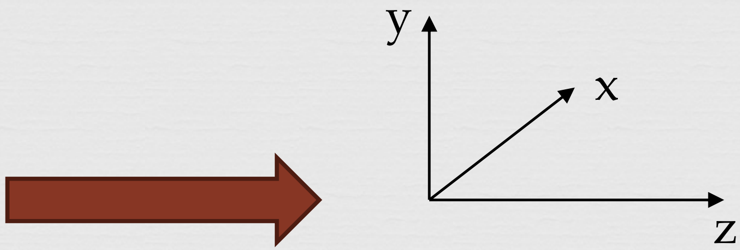
Shengwang Du

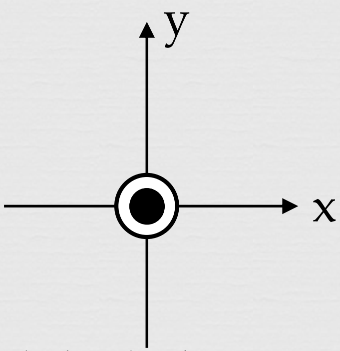
2015, the Year of Light

E Vector field

$$\vec{\mathbf{E}}(z,t) = \vec{\mathbf{E}}_x(z,t) + \vec{\mathbf{E}}_y(z,t)$$

Plane wave: $\cos(kz - \omega t + \varepsilon)$





Convention of the xyz coordinator system: Look toward the light source

For a giving a fixed z:

$$\vec{E}_x = \hat{\imath}E_{0x}\cos(-\omega t + \varphi_x) = \hat{\imath}E_{0x}\cos(\varphi_x - \omega t) = \hat{\imath}E_{0x}Re\{e^{i(\varphi_x - \omega t)}\}$$

$$\Longleftrightarrow \quad \hat{\imath}E_{0x}e^{i(\varphi_x - \omega t)} = \hat{\imath}E_{0x}e^{i\varphi_x}e^{-i\omega t}$$

$$\vec{E}_y = \hat{\imath}E_{0y}\cos(-\omega t + \varphi_y) = \hat{\imath}E_{0x}\cos(\varphi_y - \omega t) = \hat{\imath}E_{0y}Re\{e^{i(\varphi_y - \omega t)}\}$$

$$\Longleftrightarrow \quad \hat{\jmath}E_{0y}e^{i(\varphi_y - \omega t)} = \hat{\jmath}E_{0y}e^{i\varphi_y}e^{-i\omega t}$$

Jones Vectors

$$\vec{E} = \begin{bmatrix} E_{0x} e^{i\varphi_x} e^{-i\omega t} \\ E_{0y} e^{i\varphi_y} e^{-i\omega t} \end{bmatrix} = \begin{bmatrix} E_{0x} e^{i\varphi_x} \\ E_{0y} e^{i\varphi_y} \end{bmatrix} e^{-i\omega t} = \begin{bmatrix} \frac{E_{0x}}{\sqrt{E_{0x}^2 + E_{0y}^2}} e^{i\varphi_x} \\ \frac{E_{0y}}{\sqrt{E_{0x}^2 + E_{0y}^2}} e^{i\varphi_y} \end{bmatrix} \sqrt{E_{0x}^2 + E_{0y}^2}\, e^{-i\omega t}$$

Jones vector $\begin{bmatrix} E_{0x} e^{i\varphi_x} \\ E_{0y} e^{i\varphi_y} \end{bmatrix}$

Normalized Jones vector
(Polarization unit vector)

$$\begin{bmatrix} \frac{E_{0x}}{\sqrt{E_{0x}^2 + E_{0y}^2}} e^{i\varphi_x} \\ \frac{E_{0y}}{\sqrt{E_{0x}^2 + E_{0y}^2}} e^{i\varphi_y} \end{bmatrix}$$

Operations of (complex) Jones Vectors

Both E_x and Ey can be compex numbers

Conjugation:

$$\vec{E} = \begin{bmatrix} E_x \\ E_y \end{bmatrix} \qquad \vec{E}^+ = [E_x^* \quad E_y^*] \qquad \left|\vec{E}\right|^2 = \vec{E}^+ \cdot \vec{E} = [E_x^* \quad E_y^*] \begin{bmatrix} E_x \\ E_y \end{bmatrix} = E_x^* E_x + E_y^* E_y$$

The same polarization

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} \Longleftrightarrow A \begin{bmatrix} E_x \\ E_y \end{bmatrix}$$

Normalized Jones vector
(Polarization unit vector)

$$\vec{E}_P = \begin{bmatrix} \dfrac{E_{0x}}{\sqrt{E_{0x}^2 + E_{0y}^2}} e^{i\varphi_x} \\ \dfrac{E_{0y}}{\sqrt{E_{0x}^2 + E_{0y}^2}} e^{i\varphi_y} \end{bmatrix} \Longrightarrow \left|\vec{E}_P\right|^2 = \vec{E}_P{}^+ \cdot \vec{E}_P = 1$$

Polarization Unit Vector

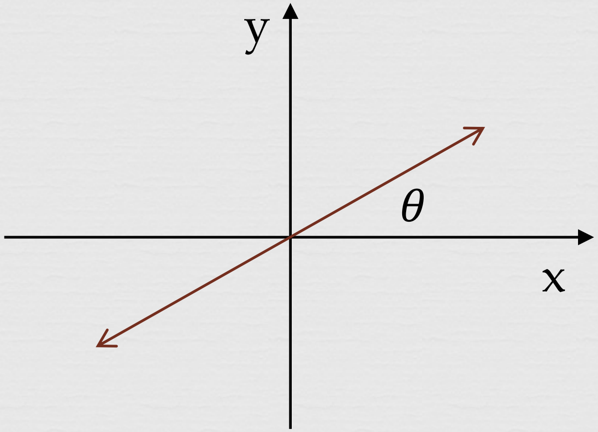
$$\vec{E}_P = \begin{bmatrix} \frac{E_{0x}}{\sqrt{E_{0x}^2+E_{0y}^2}} e^{i\varphi_x} \\ \frac{E_{0y}}{\sqrt{E_{0x}^2+E_{0y}^2}} e^{i\varphi_y} \end{bmatrix} = e^{i\varphi_x} \begin{bmatrix} \frac{E_{0x}}{\sqrt{E_{0x}^2+E_{0y}^2}} \\ \frac{E_{0y}}{\sqrt{E_{0x}^2+E_{0y}^2}} e^{i(\varphi_y-\varphi_x)} \end{bmatrix}$$

The same polarization $\Longleftrightarrow$

$$\vec{E}_P = \begin{bmatrix} \frac{E_{0x}}{\sqrt{E_{0x}^2+E_{0y}^2}} \\ \frac{E_{0y}}{\sqrt{E_{0x}^2+E_{0y}^2}} e^{i(\varphi_y-\varphi_x)} \end{bmatrix} = \begin{bmatrix} \frac{E_{0x}}{\sqrt{E_{0x}^2+E_{0y}^2}} \\ \frac{E_{0y}}{\sqrt{E_{0x}^2+E_{0y}^2}} e^{i\Delta\varphi} \end{bmatrix}$$

$$\vec{E}_P = \begin{bmatrix} \cos\theta \\ \sin\theta \, e^{i\Delta\varphi} \end{bmatrix}$$

Linear Polarization $\Delta\varphi = 0$



$$\vec{E}_x = \hat{\imath} E_{0x} e^{i(kz-\omega t)}$$

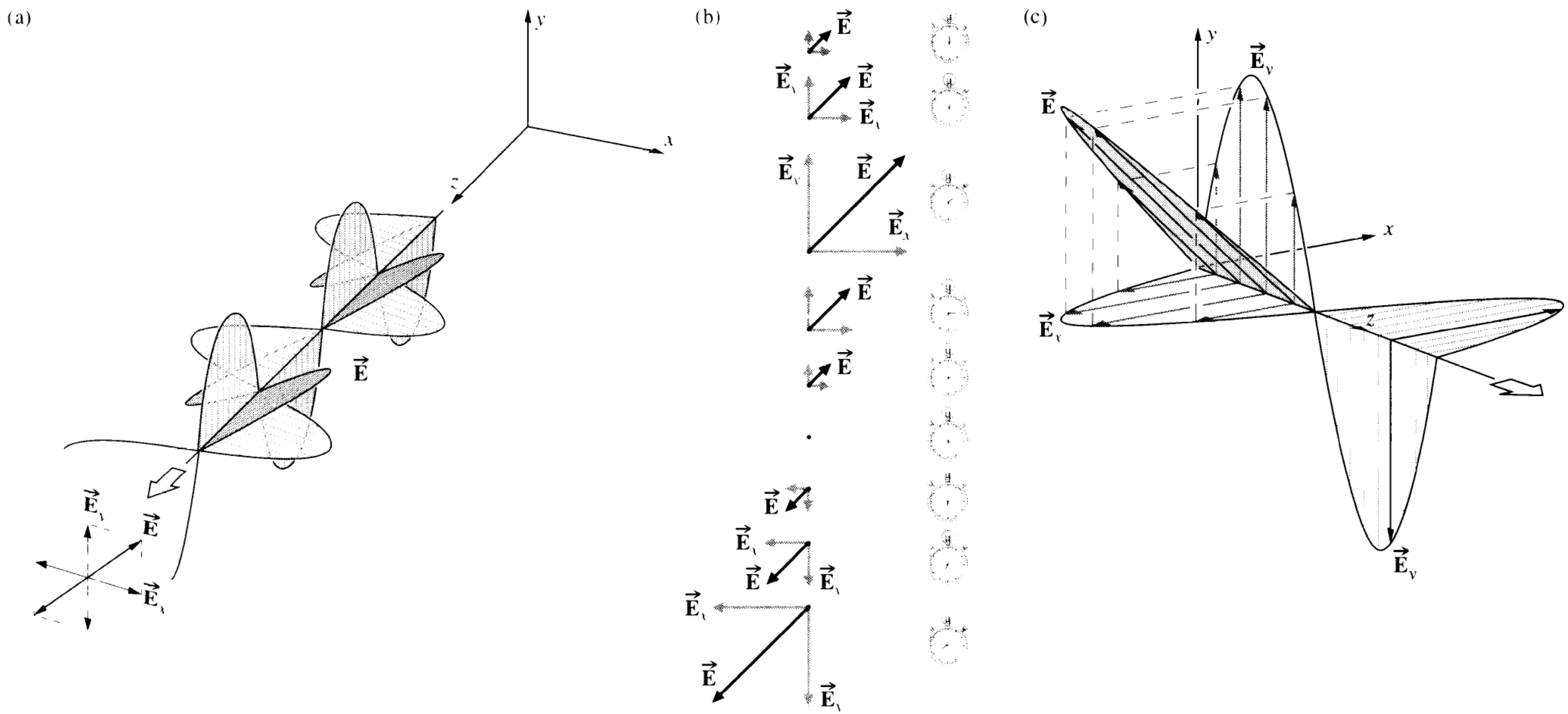
$$\vec{E}_y = \hat{\jmath} E_{0y} e^{i(kz-\omega t)}$$

$E_{0x} = E_0 \cos\theta$

$E_{0y} = E_0 \sin\theta$

$$\vec{E}_{l\theta} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

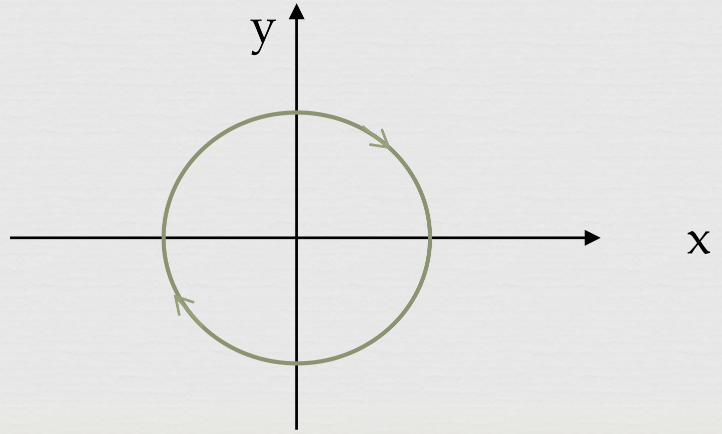
Linear Polarization $\Delta\varphi = 0$



Right Circular Polarization

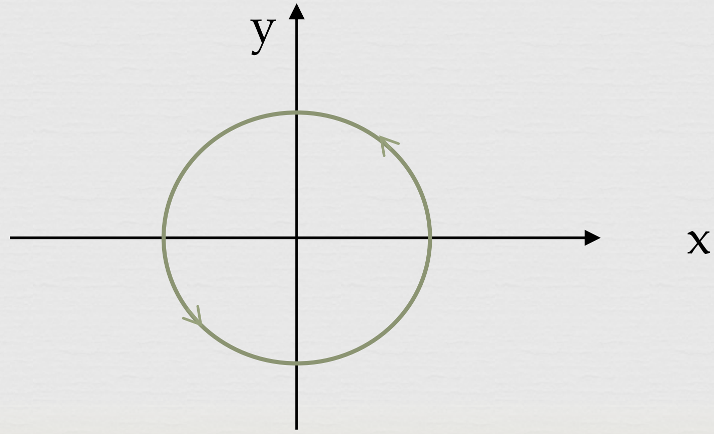
$$E_{0x}=E_{0y}=E_0, \Delta\varphi = -\frac{\pi}{2}$$

$$\vec{E}_{\mathcal{R}} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} e^{-i\frac{\pi}{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ e^{-i\frac{\pi}{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$



Left Circular Polarization $E_{0x}=E_{0y}=E_0, \Delta\varphi = \frac{\pi}{2}$

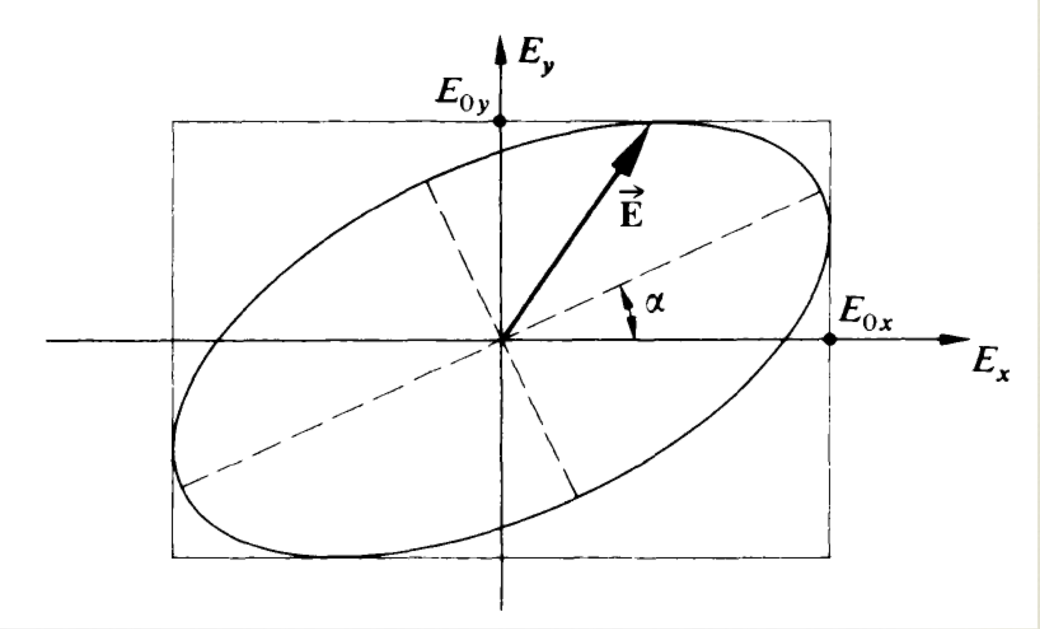
$$\vec{E}_{\mathcal{L}} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} e^{i\frac{\pi}{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ e^{i\frac{\pi}{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}$$



Elliptical Polarization

$E_{0x} \neq E_{0y}$ $\Delta\varphi \neq 0$

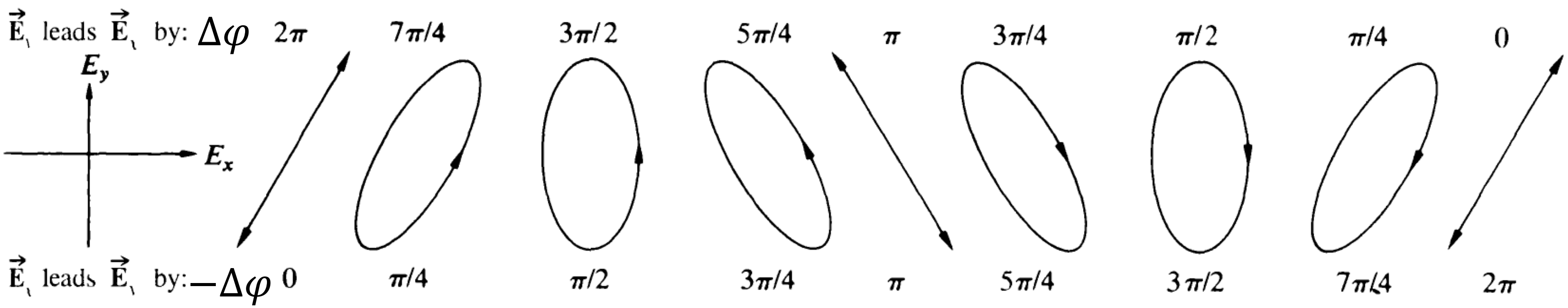
$$\vec{E}_P = \begin{bmatrix} \cos\theta \\ \sin\theta \, e^{i\Delta\varphi} \end{bmatrix}$$



$$\tan 2\alpha = \tan 2\theta \cos \Delta\varphi$$

Polarizations

$$\vec{E}_P = \begin{bmatrix} \cos\theta \\ \sin\theta\, e^{i\Delta\varphi} \end{bmatrix}$$



(a)

Polarizers

- Linear polarizer
- Circular polarizer
- Elliptical polarizer

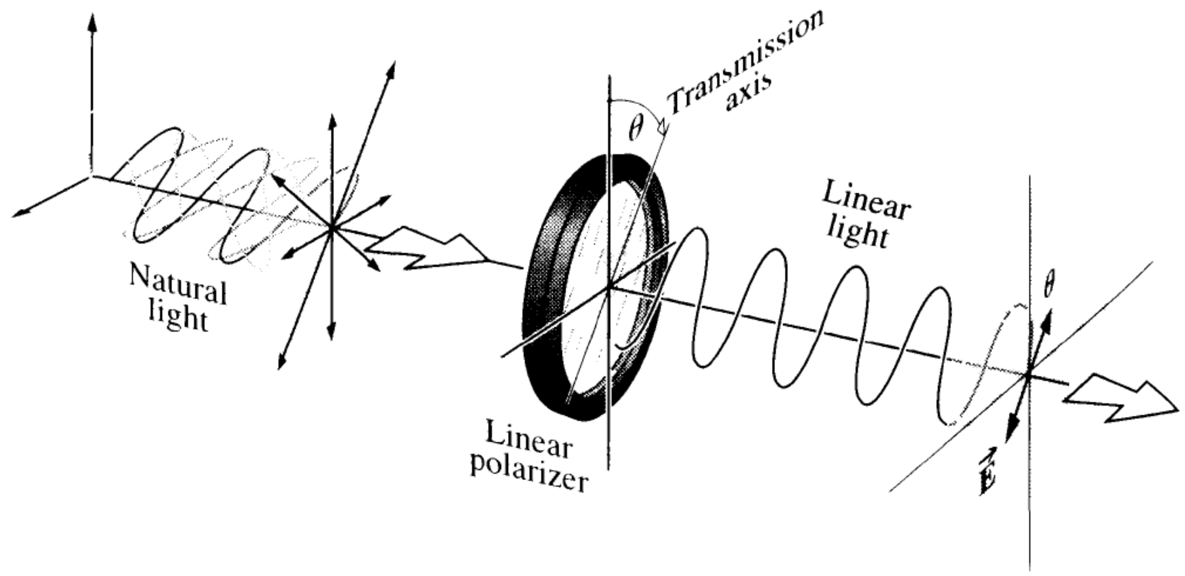
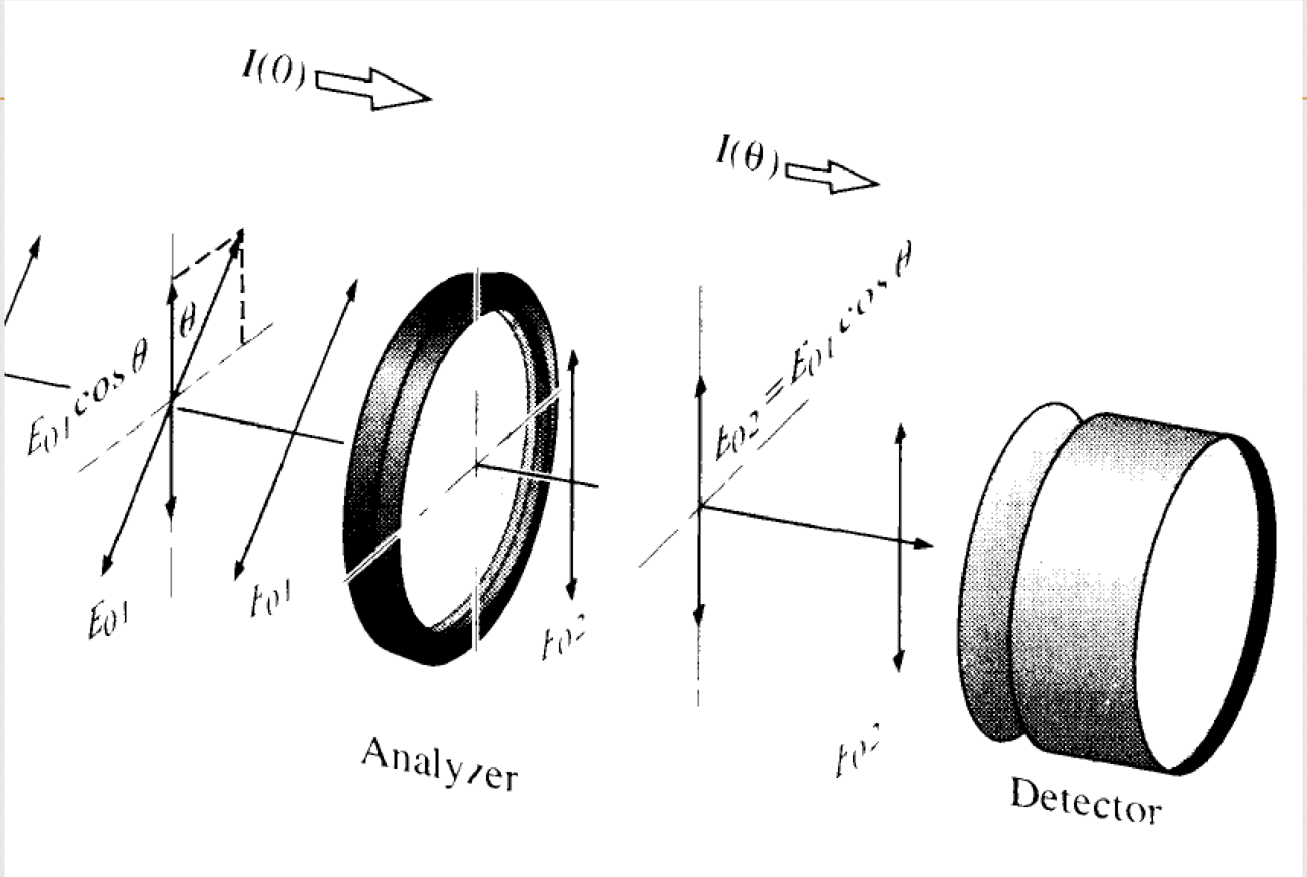


Figure 8.10 Natural light incident on a linear polarizer tilted at an angle θ with respect to the vertical.

Linear Polarizer



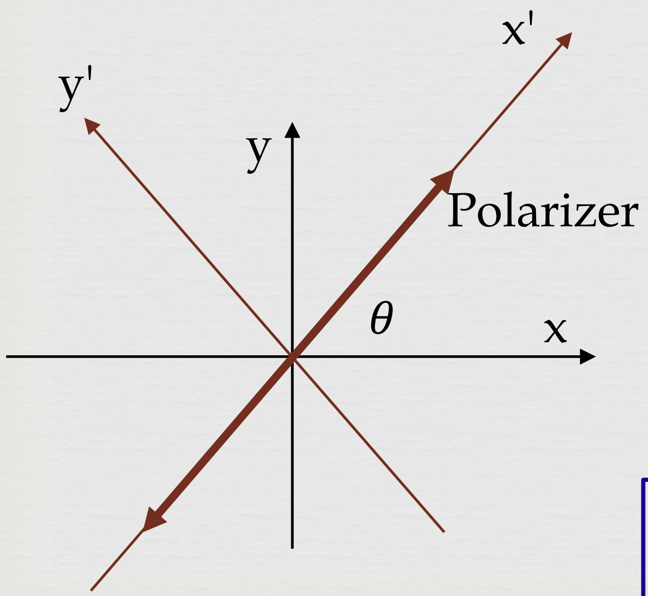
$$I(\theta) = \frac{c\epsilon_0}{2} E_{01}^2 \cos^2\theta$$

Linear Polarizer

input $\vec{E}_{in} = \begin{bmatrix} E_{xin} \\ E_{yin} \end{bmatrix}$

What is the output?

Output (complex) amplitude $E_{out} = E_{xin}\cos\theta + E_{yin}\sin\theta$



x'-y' coordinators: $\vec{E}_{out} = E_{out}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

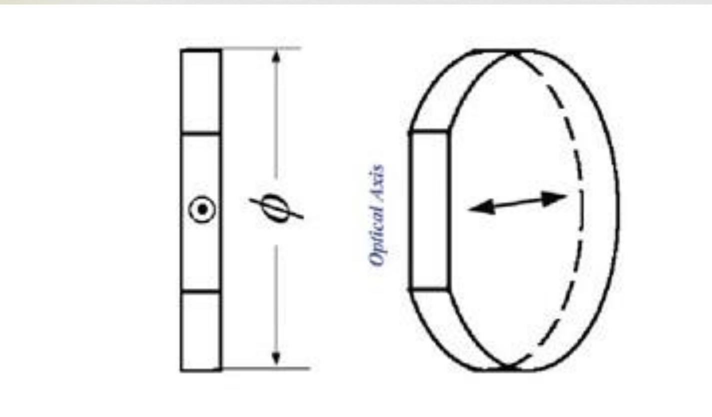
$$= (E_{xin}\cos\theta + E_{yin}\sin\theta)\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

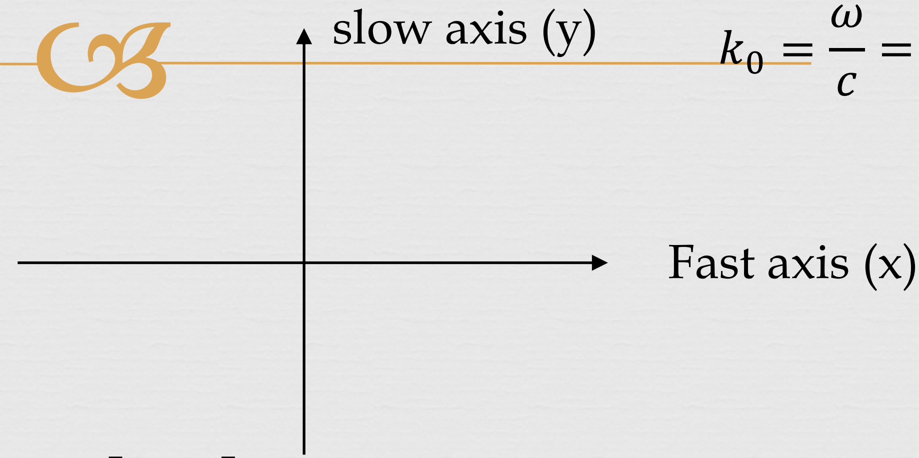
x-y coordinators: $\vec{E}_{out} = E_{out}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

$$= (E_{xin}\cos\theta + E_{yin}\sin\theta)\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$\boxed{\mathcal{L}(\theta) = \begin{bmatrix} cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & sin^2\theta \end{bmatrix}}$$

Retarders (Wave Plates)





$$k_0 = \frac{\omega}{c} = \frac{2\pi}{\lambda_0}$$

$$n_s > n_f$$

$$k_s = n_s k_0 > k_f = n_f k_0$$

$$\Delta\varphi_{fs} = (k_s - k_f)d = \frac{2\pi}{\lambda_0}\mathrm{d}(n_s - n_f)$$

$$\Delta\varphi_{fs} = \frac{1}{\#} \times 2\pi \quad 1/\#\ \text{Wave plate}$$

$$\vec{E}_{in} = \begin{bmatrix} E_{xin} \\ E_{yin} \end{bmatrix} \longrightarrow \vec{E}_{out} = \begin{bmatrix} E_{xin} \\ E_{yin} e^{i\Delta\varphi_{fs}} \end{bmatrix}$$

$$W_{1/\#} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\Delta\varphi_{fs}} \end{bmatrix} = e^{i\frac{\Delta\varphi_{fs}}{2}} \begin{bmatrix} e^{-i\frac{\Delta\varphi_{fs}}{2}} & 0 \\ 0 & e^{i\frac{\Delta\varphi_{fs}}{2}} \end{bmatrix}$$

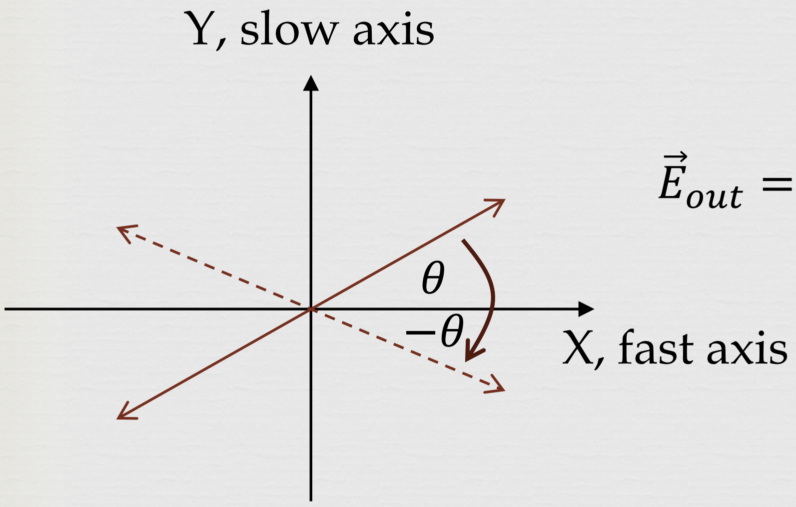
$$\boxed{W_{1/\#} \triangleq \begin{bmatrix} e^{-i\frac{\Delta\varphi_{fs}}{2}} & 0 \\ 0 & e^{i\frac{\Delta\varphi_{fs}}{2}} \end{bmatrix}}$$

Half-Wave Plate

$$\Delta\varphi_{fs} = \frac{1}{2} \times 2\pi + 2m\,\pi = \pi + 2m\,\pi$$

$$\boxed{W_{1/2} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}}$$

Input (linear) $\vec{E}_{in} = \begin{bmatrix} E_{xin} \\ E_{yin} \end{bmatrix} = E_0 \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$



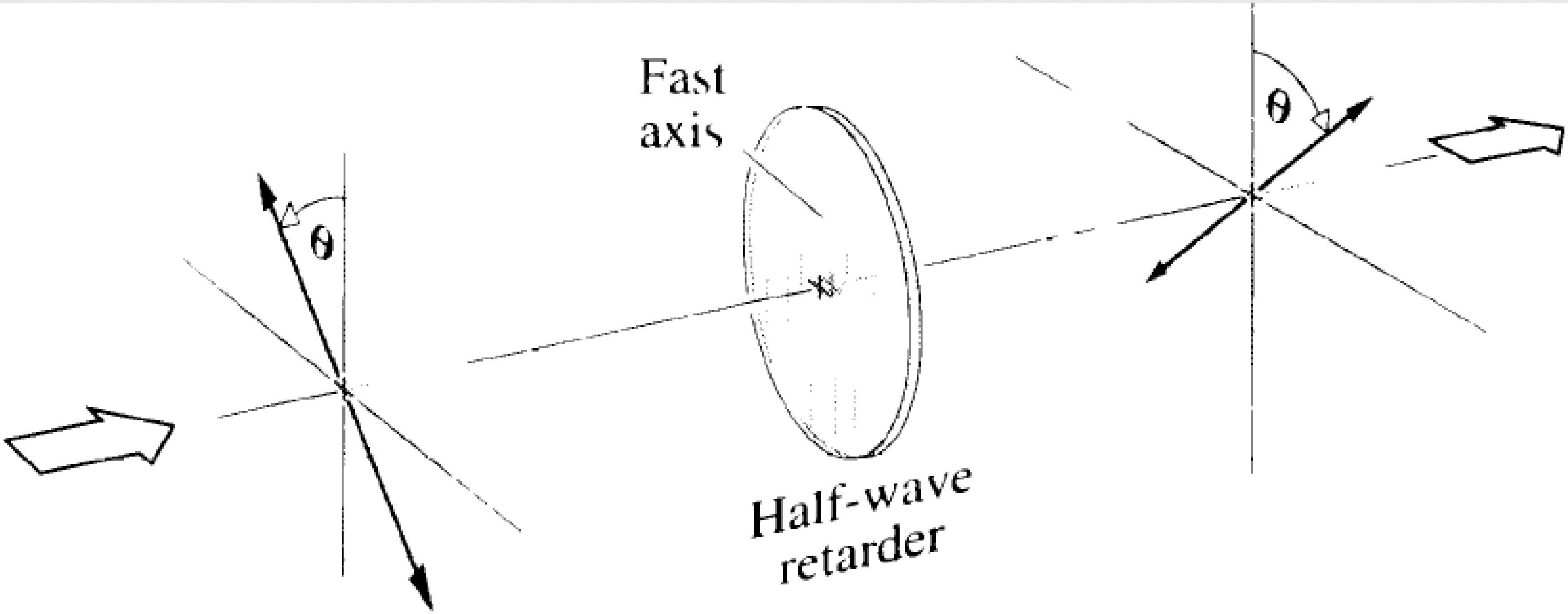
Output

$$\vec{E}_{out} = W_{1/2}\vec{E}_{in} = E_0 \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = -iE_0 \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix}$$

$$= -iE_0 \begin{bmatrix} \cos(-\theta) \\ \sin(-\theta) \end{bmatrix}$$

"Reflection" according to the fast axis

Half-Wave Plate



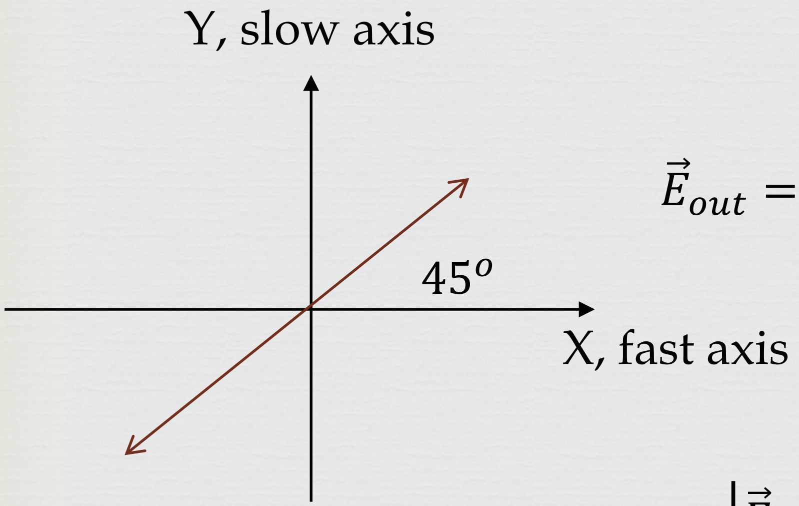
Quarter-Wave Plate

$$\Delta\varphi_{fs} = \frac{1}{4} \times 2\pi + 2\mathrm{m}\,\pi = \frac{\pi}{2} + 2\mathrm{m}\,\pi$$

m-order

$$\boxed{W_{1/4} = \begin{bmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} = e^{-i\pi/4} \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}}$$

Input $\vec{E}_{in} = \frac{E_0}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ Linearly polarized



Output

$$\vec{E}_{out} = W_{1/4}\vec{E}_{in} = \frac{E_0}{\sqrt{2}} e^{-i\pi/4} \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{E_0}{\sqrt{2}} e^{-i\pi/4} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Left circularly polarized

$$\left|\vec{E}_{out}\right| = \sqrt{\vec{E}_{out}^{\,+} \cdot \vec{E}_{out}} = |E_0|$$

Linear polarization ➔ Circular polarization

Quarter-Wave Plate

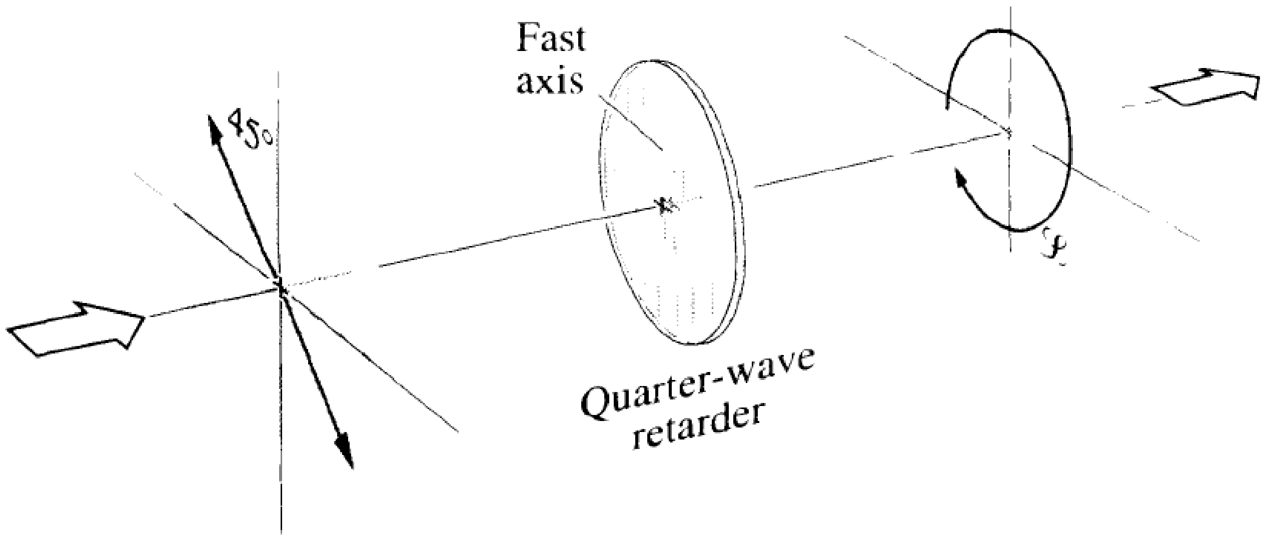
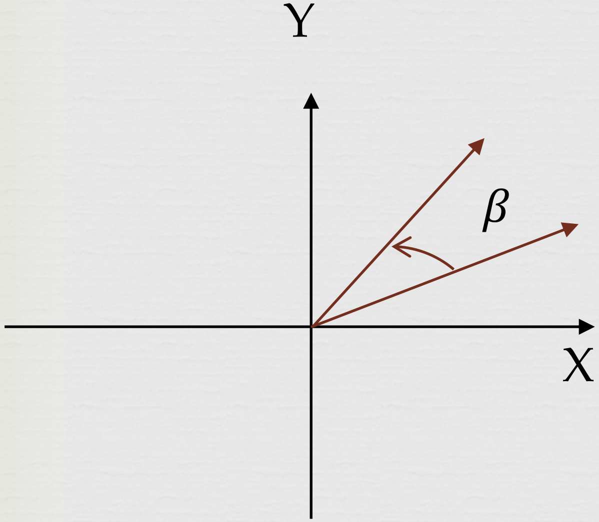


Figure 8.40 A quarter-wave plate transforms light initially linearly polarized at an angle 45° (oscillating in the first and third quadrants) into left circular light (rotating counterclockwise looking toward the source).

Rotation Operator



$$R(\beta) = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

$$R^{-1}(\beta) = R(-\beta) = \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix}$$

Linear Polarizer

$$\boxed{\mathcal{L}(\theta) = \begin{bmatrix} cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & sin^2\theta \end{bmatrix}}$$

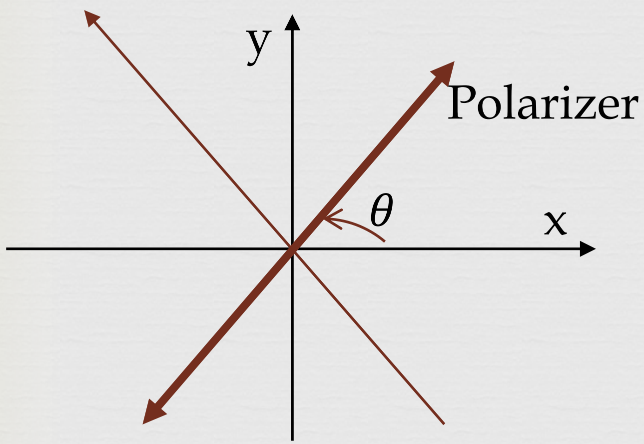
$$\mathcal{L}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathcal{L}(\theta) = R(\theta)\mathcal{L}(0)R(-\theta)$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin\theta\sin\theta \end{bmatrix}$$

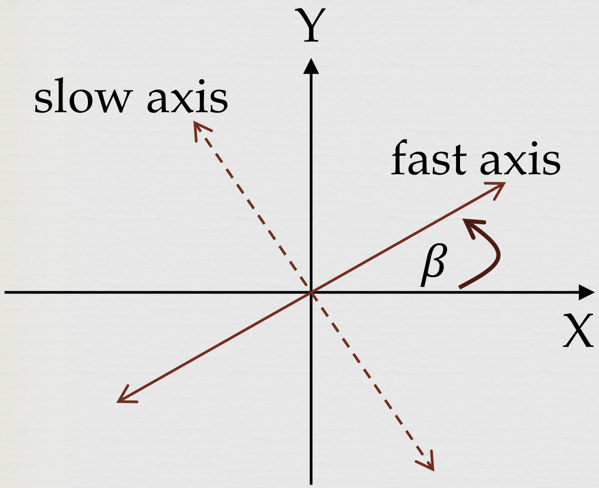


Quarter-Wave Plate

$\Delta\varphi_{fs} = \frac{1}{2} \times 2\pi + 2m\,\pi = \pi + 2m\,\pi$

$$\boxed{W_{1/2} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}}$$

$$W_{\frac{1}{2}}(\beta) = R(\beta) \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \mathrm{R}(-\beta)$$



Example: $\beta=\pi/2$

$$R(\pi/2) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

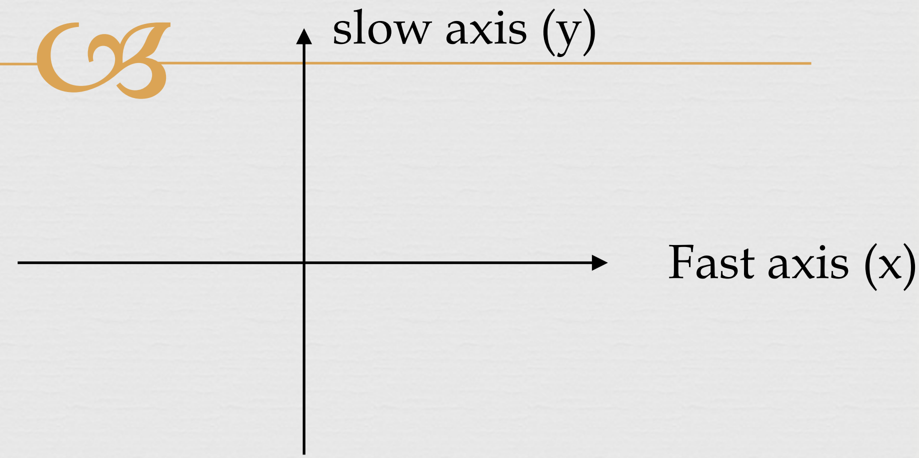
$$W_{\frac{1}{2}}(\pi/2) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

Manipulating Polarization

- Combining a half-wave plate and quarter-wave plate, one can change a polarization to an arbitrary polarization.
- Combining a half-wave plate, quarter-wave plate, and a linear polarizer, one can construct an arbitrary polarizer.

Retarders : Spin ½ Rotation (not required)

$$W_{1/\#} \triangleq \begin{bmatrix} e^{-i\frac{\Delta\varphi_{fs}}{2}} & 0 \\ 0 & e^{i\frac{\Delta\varphi_{fs}}{2}} \end{bmatrix}$$

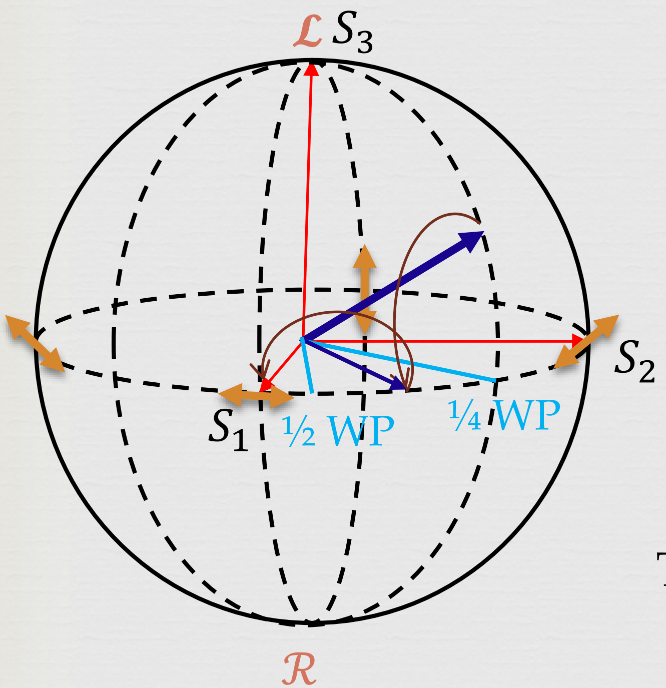


Quantum mechanics: Spin ½ system

$$R_x(\Delta\varphi_{fs}) = e^{-iS_x\Delta\varphi_{fs}/\hbar}$$

Spin 1/2	Polarization
½ spin two-component internal space	2D polarization space (Jones vector)
3D real spatial space	3D Stokes vector space
3D rotation operator Matrix	Mueller Matrix

3D Stokes Vector and Poncare Sphere (not required)



$$\vec{E}_P = \begin{bmatrix} \cos\theta \\ \sin\theta\, e^{i\Delta\varphi} \end{bmatrix}$$

$$S_1 = \cos 2\theta$$

$$S_2 = \sin 2\theta \cos \Delta\varphi$$

$$S_2 = \sin 2\theta \sin \Delta\varphi$$

The fast axis of a wave plates:

2D Jones Space: with an angle α to the x axis

3D Stokes Space: with an angle 2α to the S1 axis and always on the S1-S2 plane

Operation of a wave plates: a right-hand rotation of $\Delta\varphi_{fs}$ along its fast axis