PHYS 3038 Optics
L10 Superposition of Waves
Reading Material: Ch7

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2015, the Year of Light

Principle of Superposition

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} \qquad [2.60]$$

sequently, if $\psi_1(\vec{\boldsymbol{r}}, t), \psi_2(\vec{\boldsymbol{r}}, t), \ldots, \psi_n(\vec{\boldsymbol{r}}, t)$ are individual solutions of Eq. (2.60), *any linear combination* of them will, in turn, be a solution. Thus

$$\psi(\vec{\boldsymbol{r}}, t) = \sum_{i=1}^{n} C_i \psi_i(\vec{\boldsymbol{r}}, t) \qquad (7.1)$$

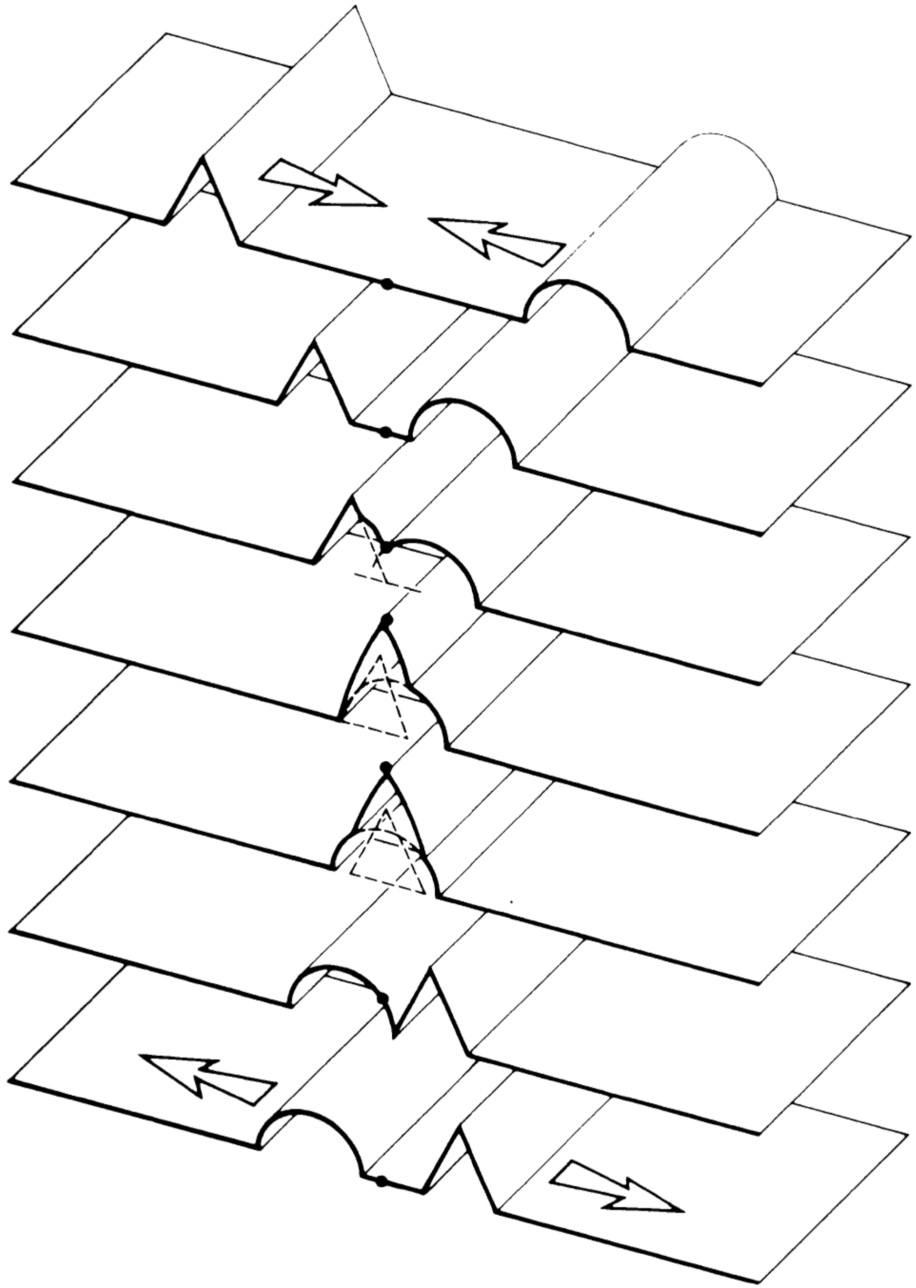


Figure 7.1 The superposition of two disturbances.

7.1 Addition of Waves of the Same Frequency

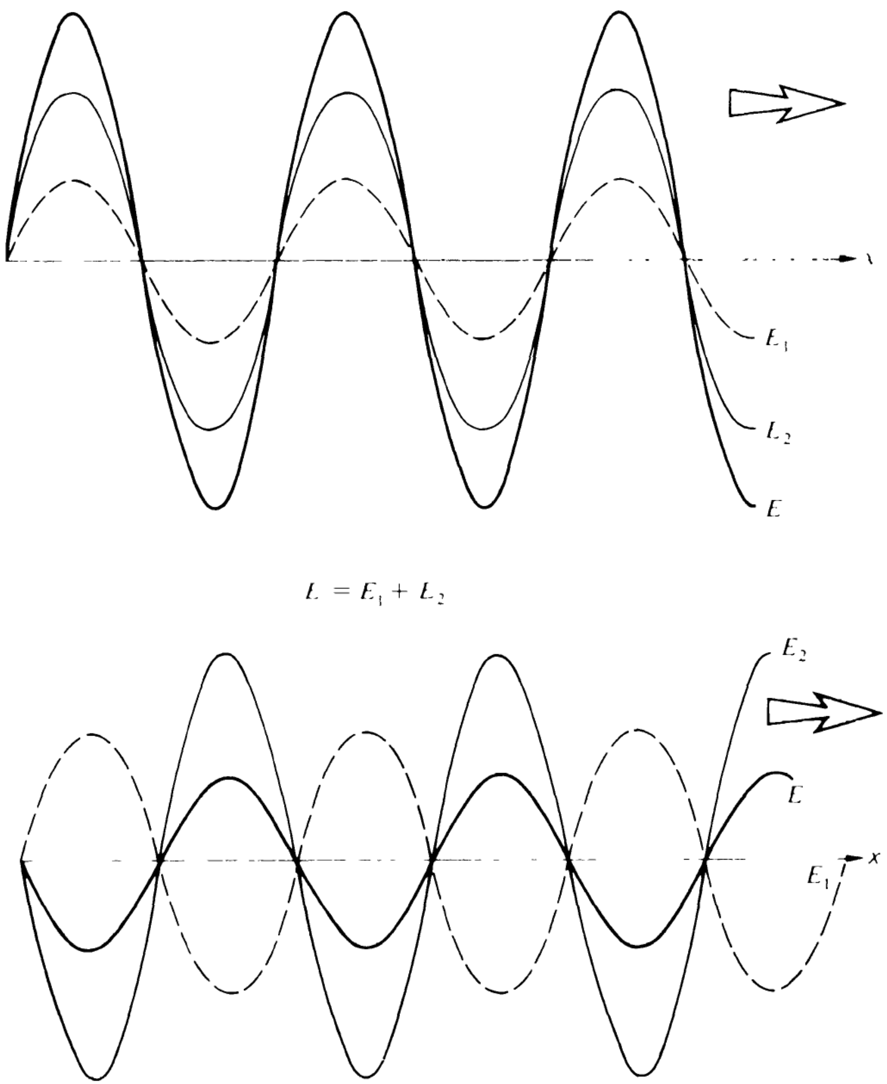


Figure 7.2 The superposition of two harmonic waves in-phase and out-of-phase.

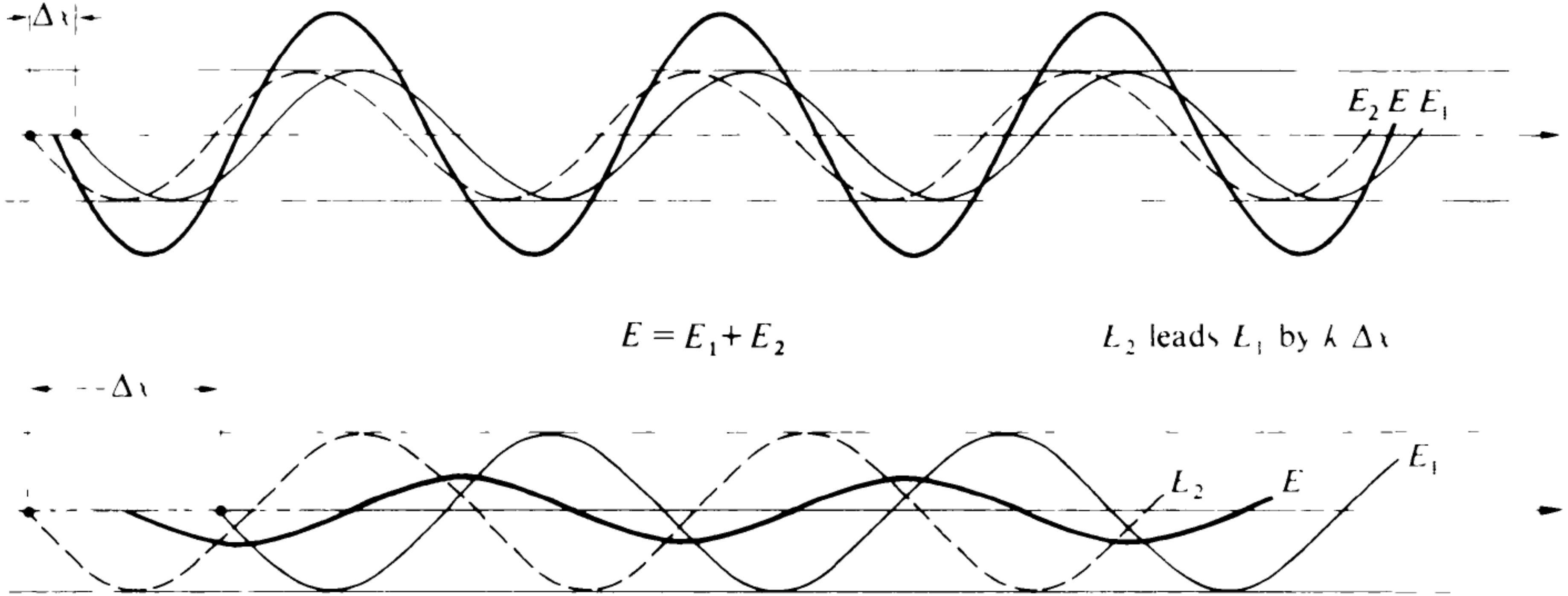
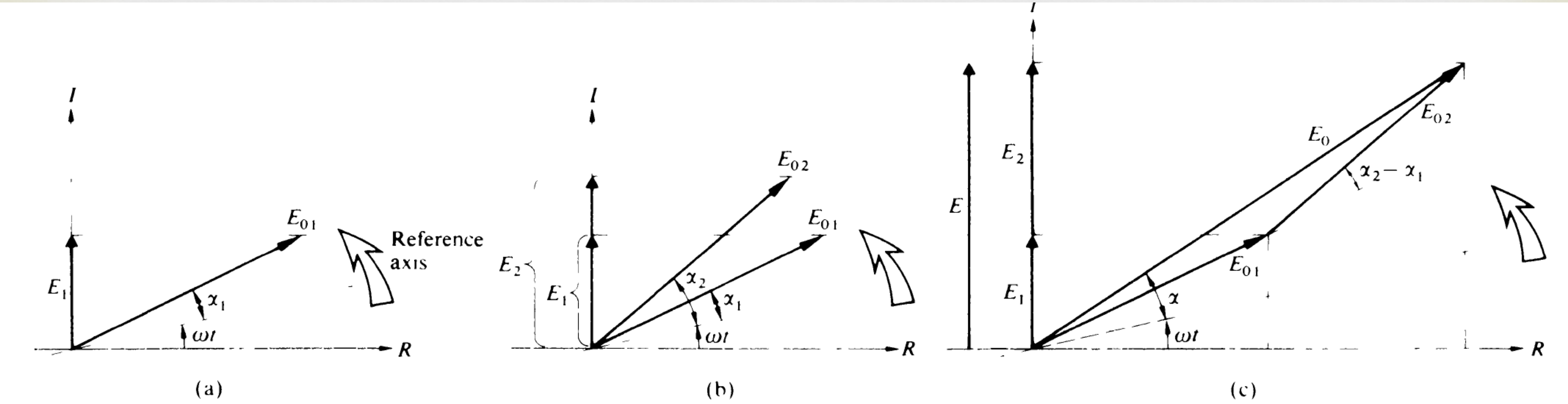


Figure 7.3 Waves out-of-phase by $k\Delta x$ radians.

Phasor Addition



$$E_1 = 5 \sin \omega t$$

$$E_2 = 10 \sin (\omega t + 45°)$$

$$E_3 = \sin (\omega t - 15°)$$

$$E_4 = 10 \sin (\omega t + 120°)$$

$$E_5 = 8 \sin (\omega t + 180°)$$

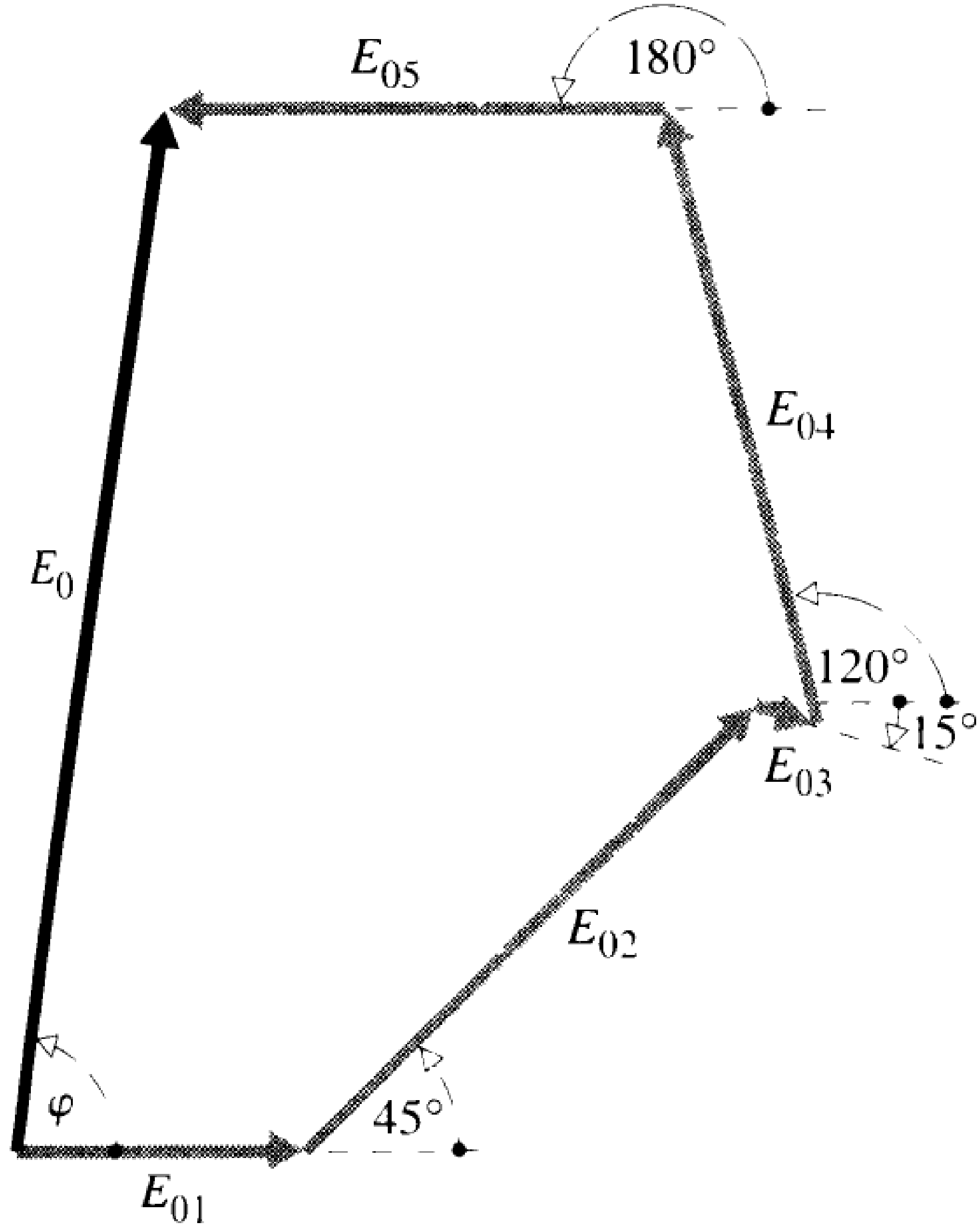
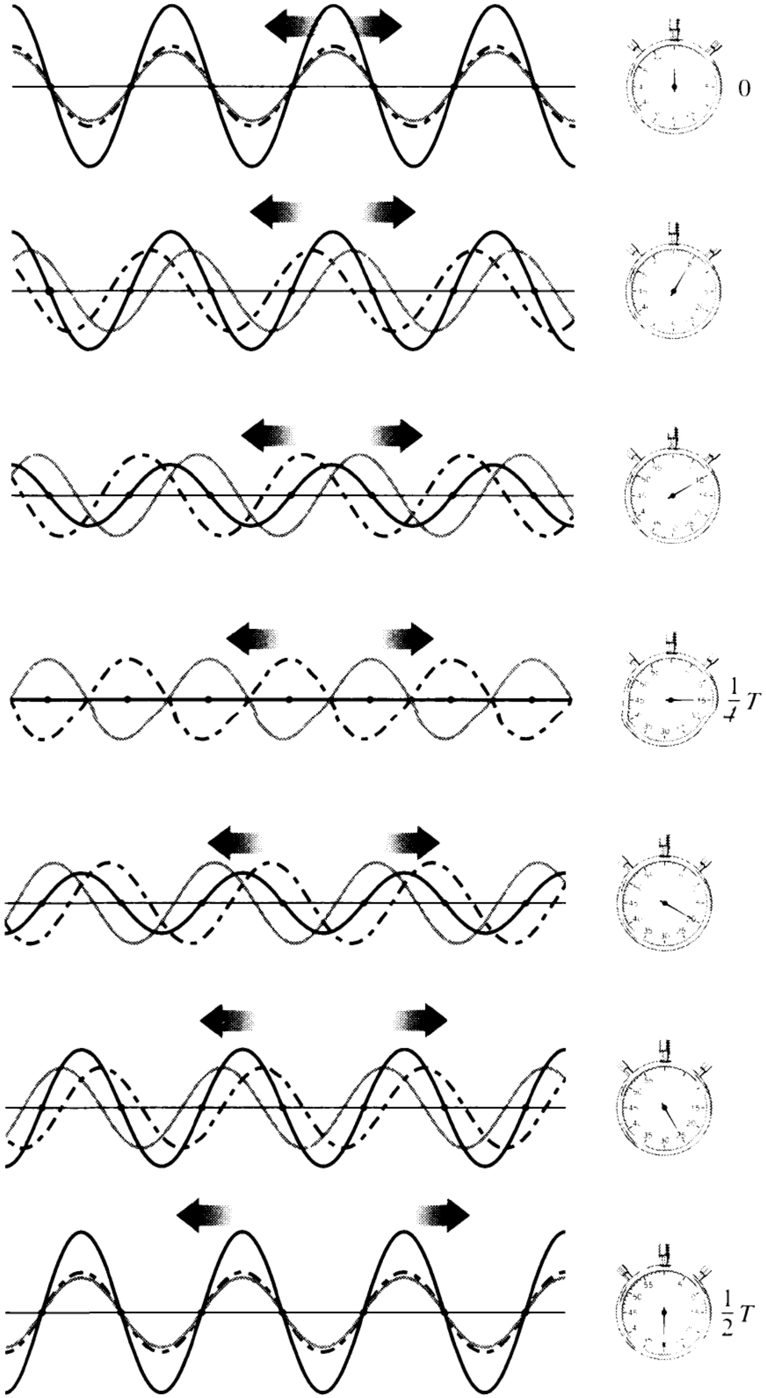
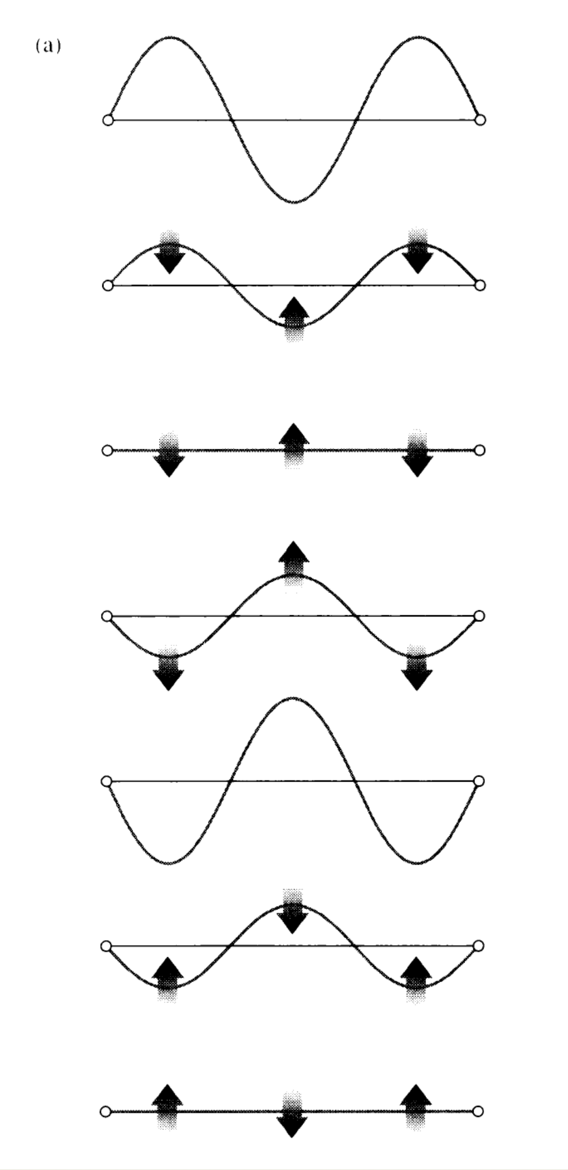


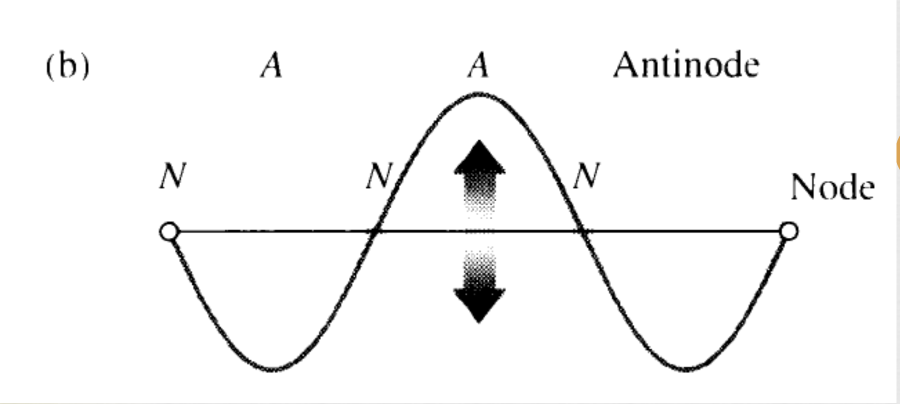
Figure 7.7 The phasor sum of E_1, E_2, E_3, E_4, and E_5.

Standing Waves





Standing Waves



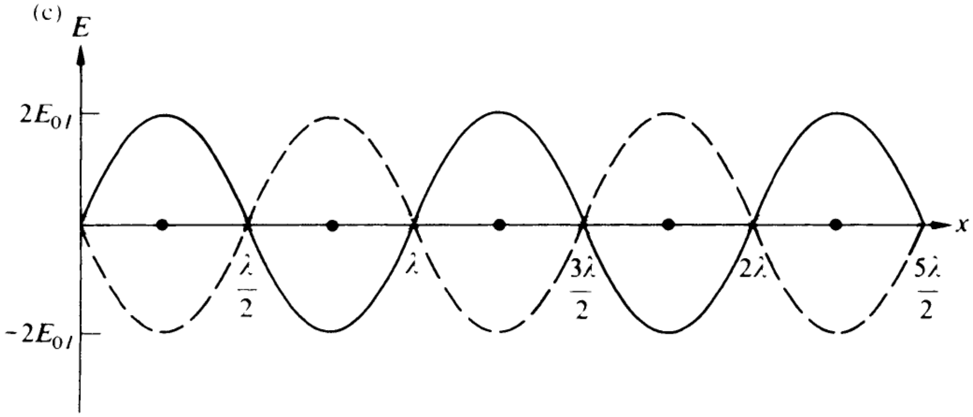
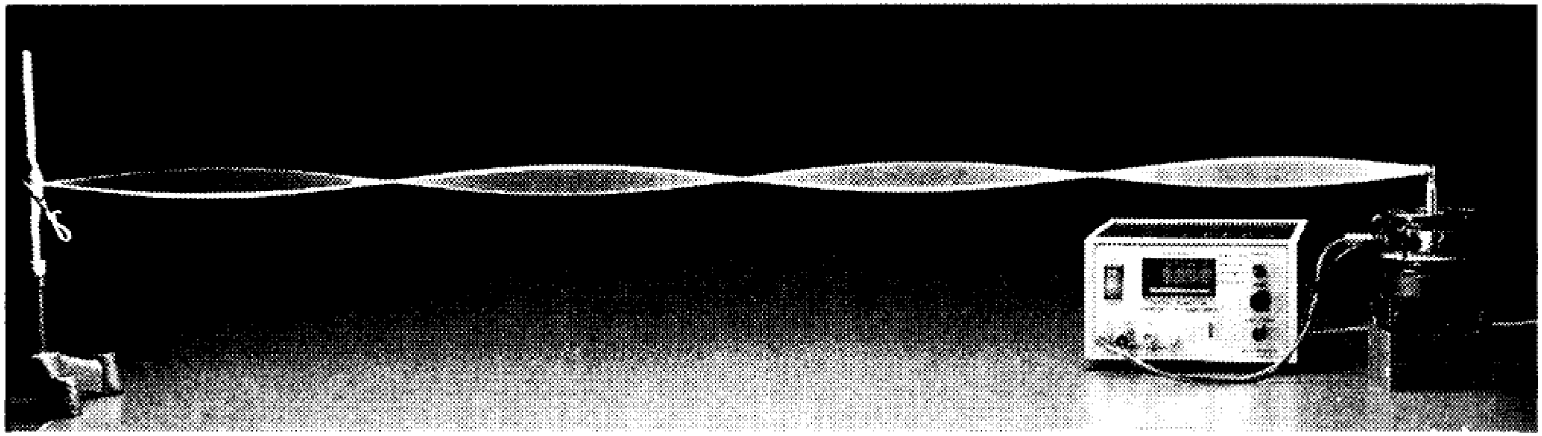
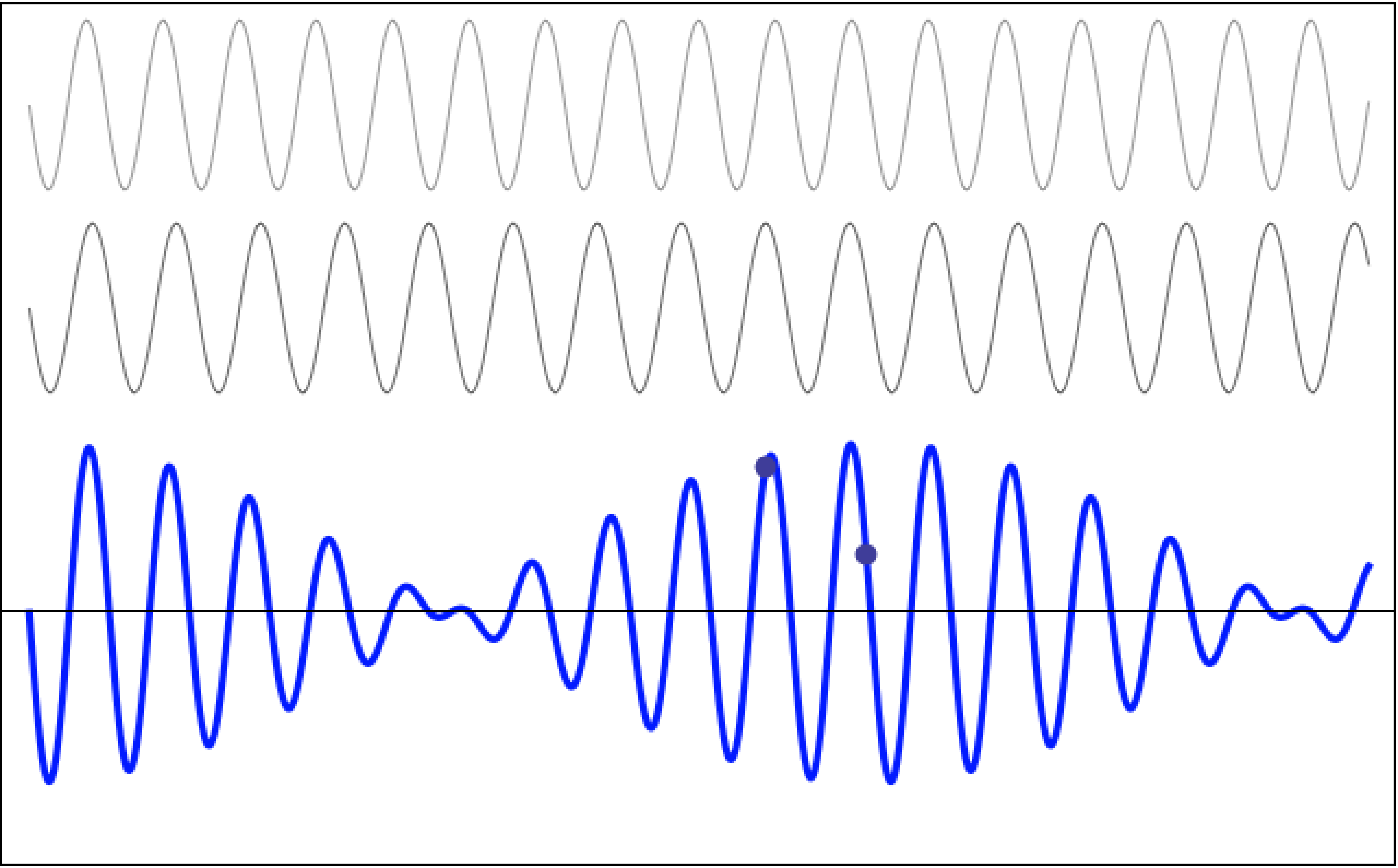


Figure 7.11 A standing wave at various times.



Standing waves on a vibrating string. (Photo courtesy PASCO.)

Beats



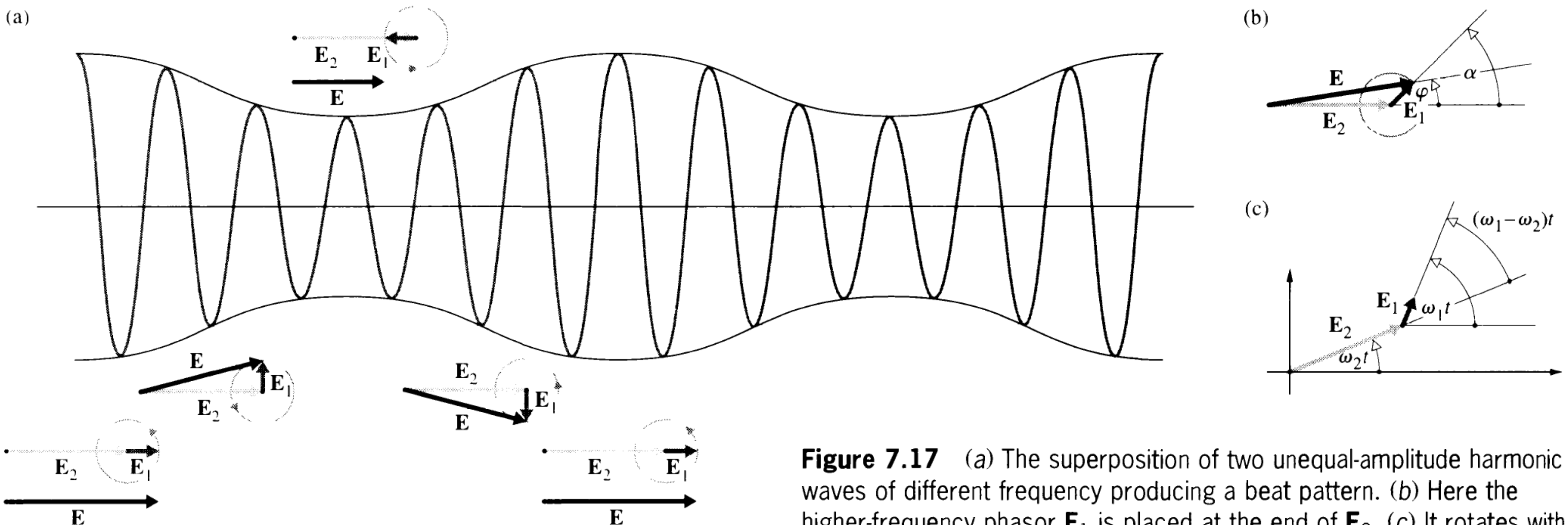


Figure 7.17 *(a)* The superposition of two unequal-amplitude harmonic waves of different frequency producing a beat pattern. *(b)* Here the higher-frequency phasor $\mathbf{E}_1$ is placed at the end of $\mathbf{E}_2$. *(c)* It rotates with the difference frequency.

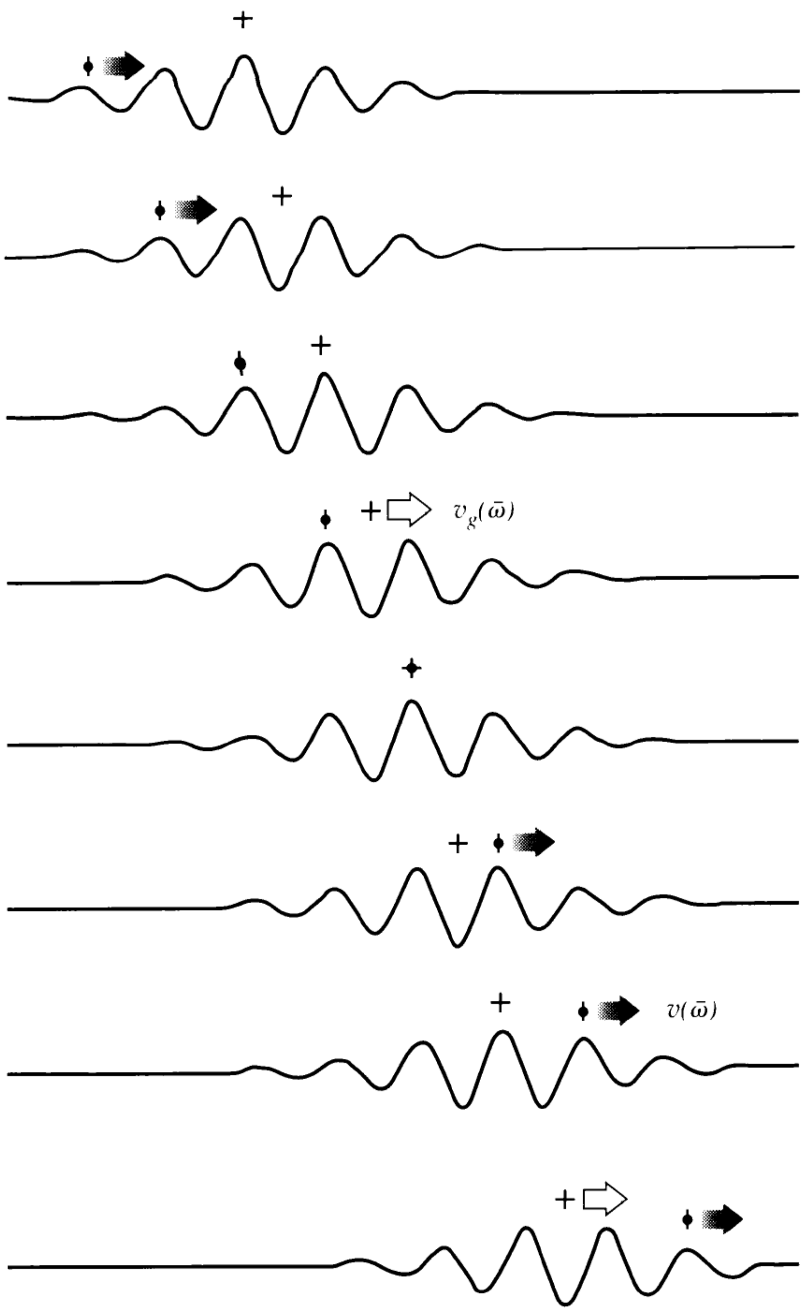


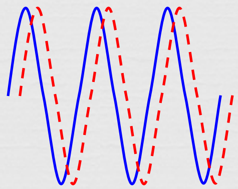
Figure 7.18 A wave pulse in a dispersive medium.

Group Velocity

$$v_g = \left(\frac{d\omega}{dk}\right)_{\bar{\omega}}$$

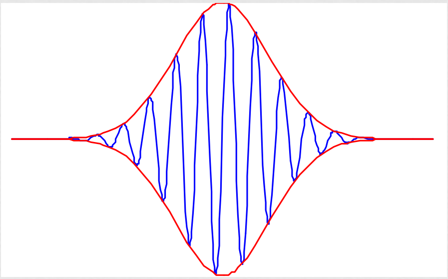
Velocities of Light in Medium

Phase Velocity



$$v_p = \frac{\omega}{k} = \frac{1}{\sqrt{\varepsilon(\omega)\mu(\omega)}}$$

Group Velocity



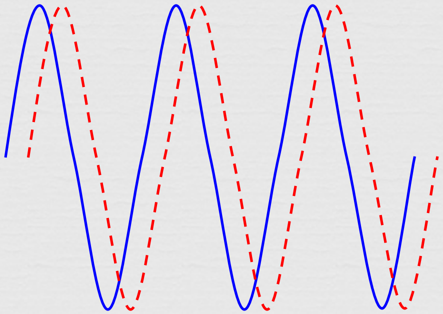
$$v_g = \frac{d\omega}{dk}$$

Information Velocity

Energy Velocity

Phase Velocity of Light

$$v_p = \frac{1}{\sqrt{\varepsilon(\omega)\mu(\omega)}} = \frac{\omega}{k} = \frac{c}{n}$$

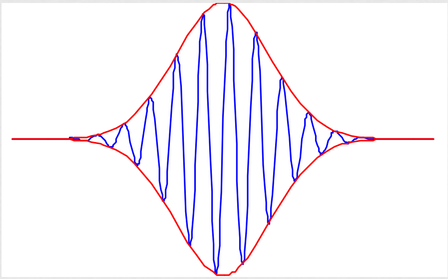


$$n = \sqrt{1+\chi}$$

- **The phase velocity of light can be :**
  - Faster than *c* (n<1)
  - Slower than *c* (n>1)

Group Velocity of Light

$$v_g = \frac{d\omega}{dk} = \frac{c}{n + \omega \dfrac{dn}{d\omega}} = \frac{c}{n_g}$$



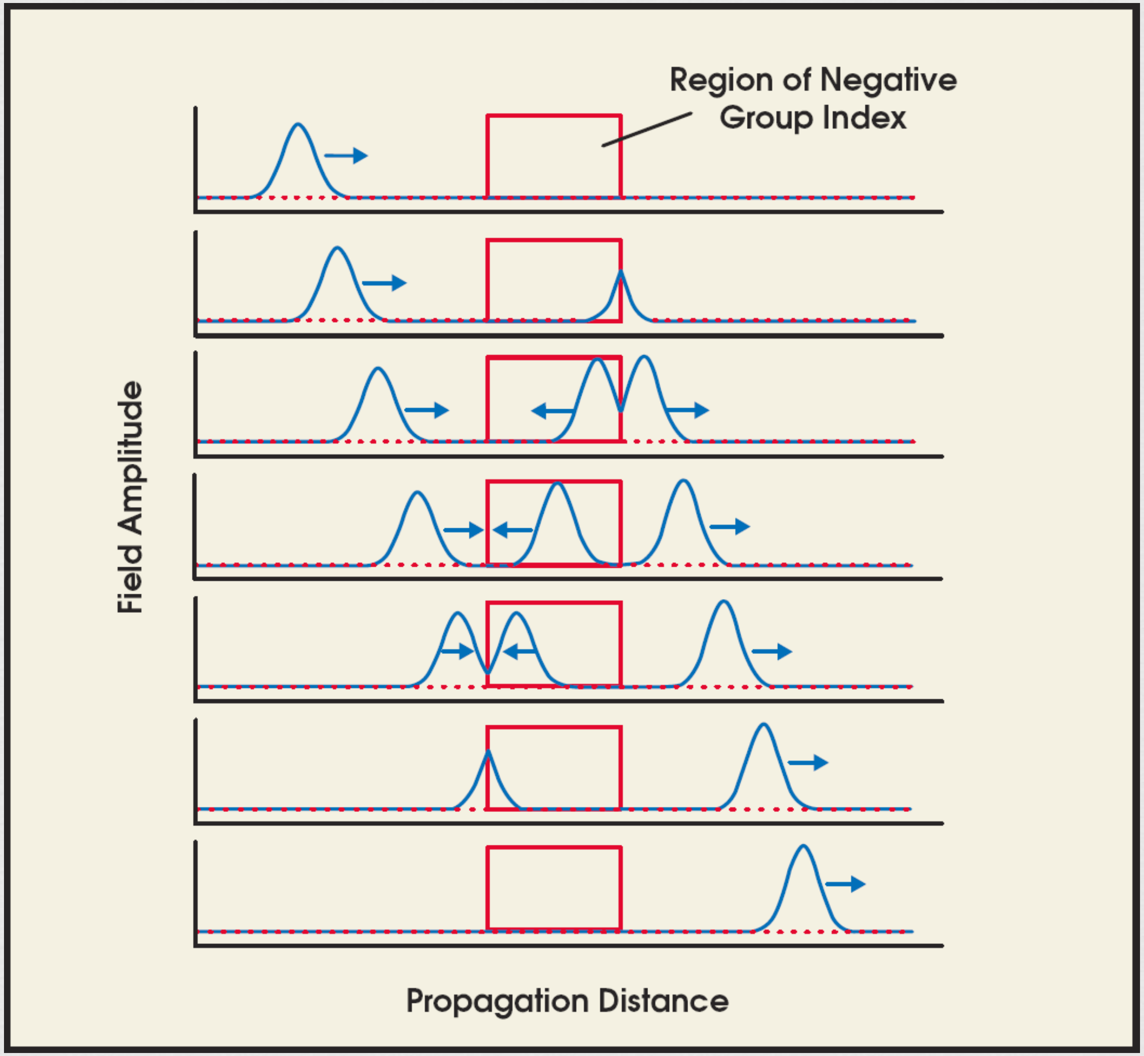
- **The group velocity of light can be :**
  - Slower than *c* (n_g>1)
  - *Faster than c (n_g<1)*
  - *Negative (n_g<0)!*

Slow Light

Fast Light (superluminal)

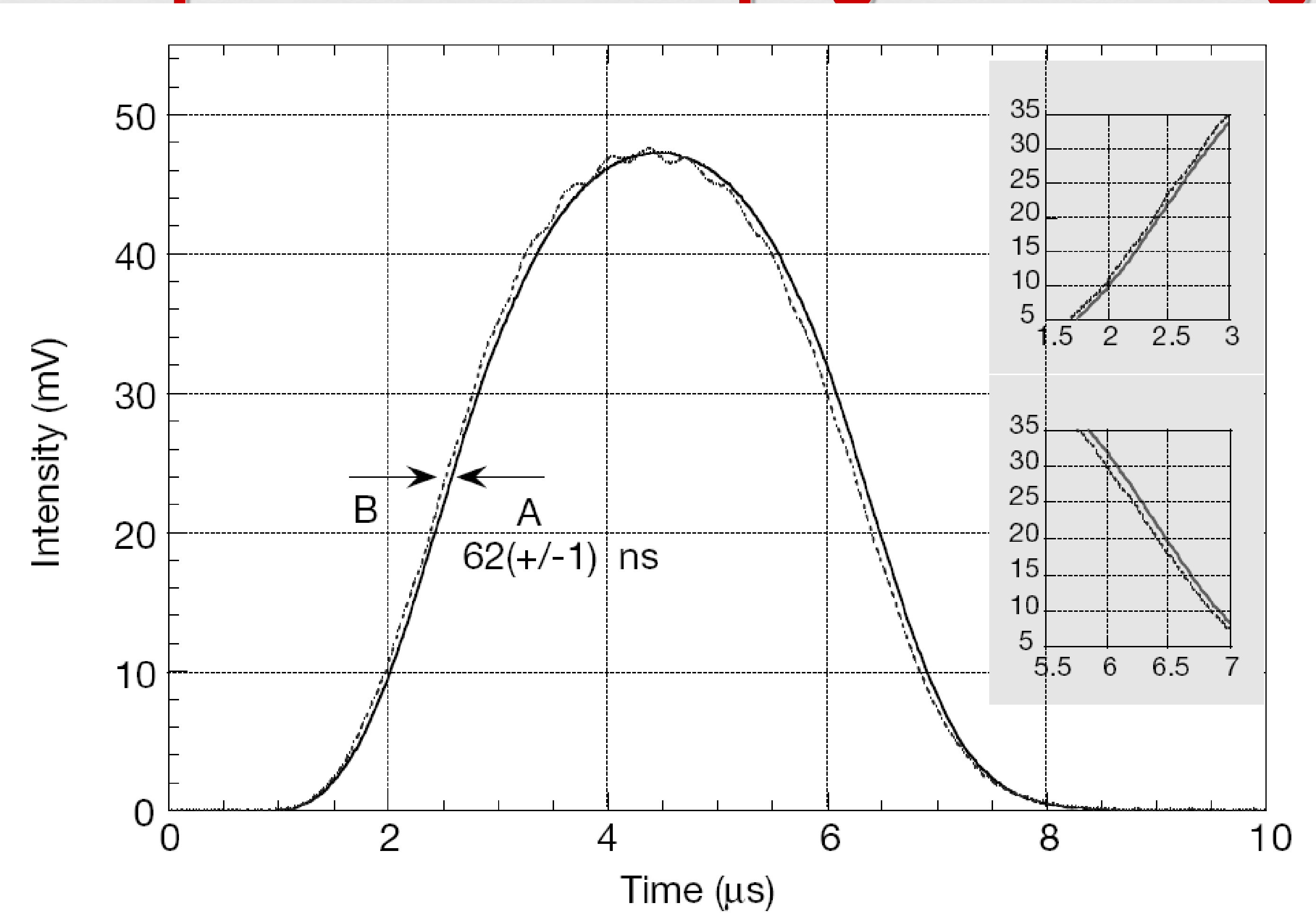
•D. J. Gauthier and R.W. Boyd, *Photonics Spectra* **82** (2007).
•G. M. Gehring, A. Schweinsberg, C. Barsi, N. Kostinski, R. W. Boyd, *Science* **312**, 895 (2006).
•L. J. Wang et. al. *Nature* **406**, 277 (2000).

Negative Group Velocity of Light



- D. J. Gauthier and R.W. Boyd, *Photonics Spectra* **82** (2007).
- G. M. Gehring, A. Schweinsberg, C. Barsi, N. Kostinski, R. W. Boyd, *Science* **312**, 895 (2006).
- L. J. Wang et. al. *Nature* **406**, 277 (2000).

Superluminal Propagation of Light



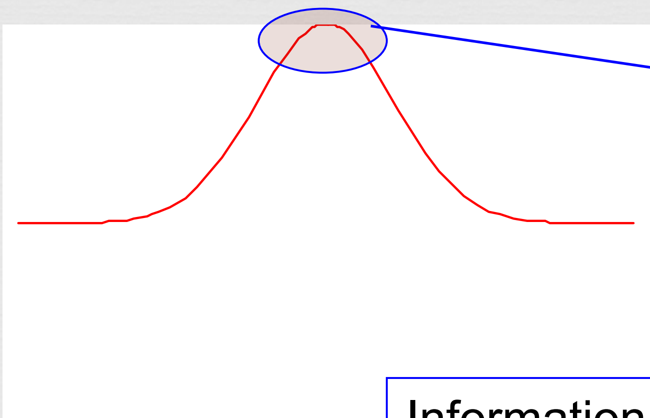
L. J. Wang, A. Kuzmich & A. Dogariu, *Nature* **406**, 277 (2000).

Topic Science: http://www.salon.com/2000/08/03/light/

Information & Information Velocity

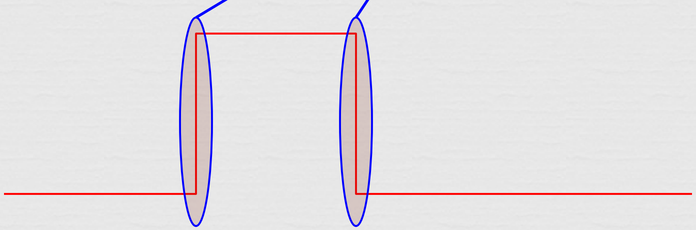
Information (entropy): measure of uncertainty

$$S(t) = k_B \ln W(t)$$

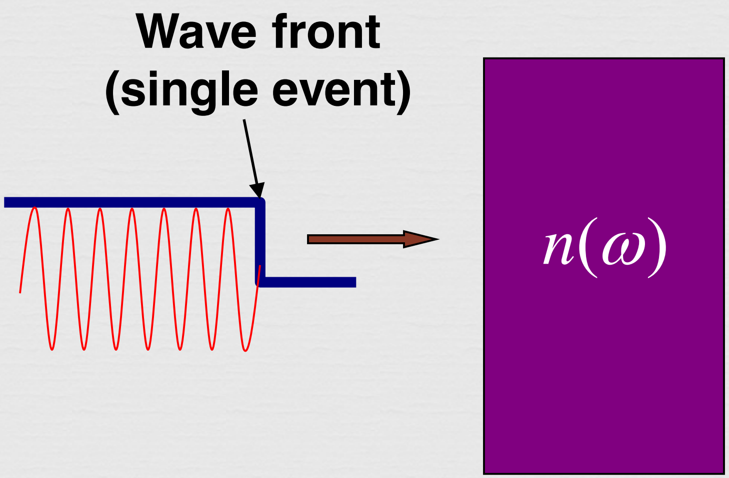


peak contains no (uncertainty) information about the future

Information velocity $\neq$ Group velocity

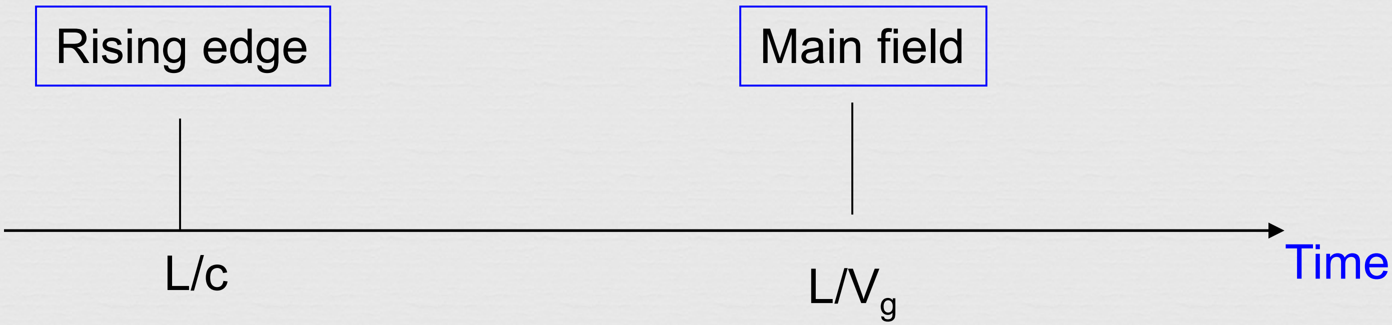


Optical Precursor & Step Pulse Propagation



Will the wave front travel faster/slower than c?

A. Sommerfeld and L. Brillouin (1914): The wave front travels exactly with c in vacuum.



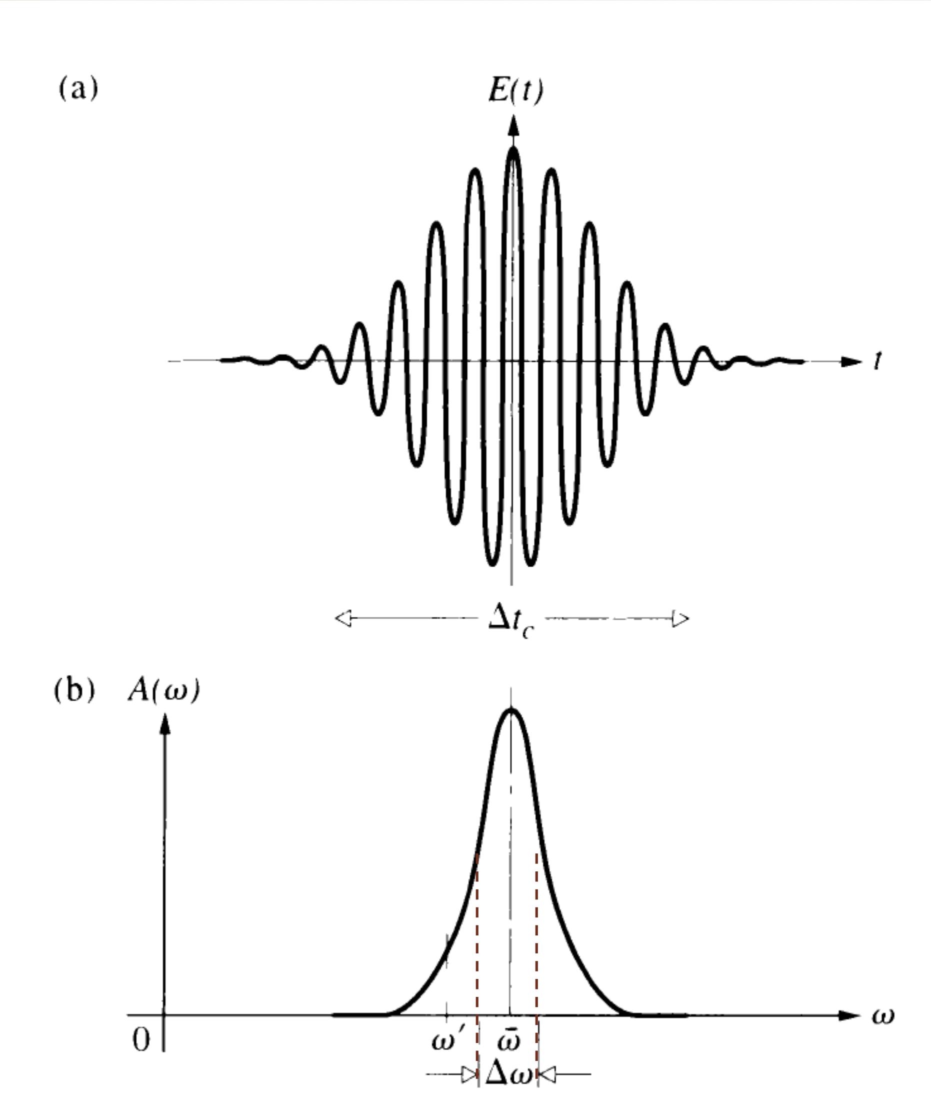
[1] J. F. Chen, H. Jeong, M. M. T. Loy, and S. Du, *Optical Precursors: From Classical Waves to Single Photons*, Springer Briefs in Physics, Springer (2013): ISBN 978-981-4451-94-9

[2] 杜胜望, "光前驱波," 物理 42, 315 (2013). [. Du, "Optical precursors," Physics 42, 315 (2013).]

[3] S. Zhang, J. F. Chen, C. Liu, M. M. T. Loy, G. K. L. Wong, and S. Du, "Optical precursor of a single photon," Phys. Rev. Lett. **106**, 243602 (2011).

[4] D. Wei, J. F. Chen, M. M. T. Loy, G. K. L. Wong, and S. Du, "Optical precursors with electromagnetically induced transparency in cold atoms," Phys. Rev. Lett. **103**, 093602 (2009).

Pulse



FWHM: Full Width at Half Maxium

$$E(z,t) = \int A(\omega)e^{i[k(\omega)z-\omega t]}d\omega$$

Bandwidth (FWHM) $\Delta\omega = 2\pi\Delta\nu$

Coherence time $\Delta t_c = 1/\Delta\nu$

Coherence length $\Delta l_c = c\Delta t_c$

Ch7. Superposition of Waves

Complex plane Wave representation

$$E(\vec{r},t) = \sum_i E_{0i}\, e^{i(\vec{k}_i\cdot\vec{r} - \omega_i t)} \quad \text{Discrete}$$

$$k(\omega) \Downarrow \omega(k)$$

$$\int E_0(\omega)\, e^{i[k(\omega)z - \omega t]}\, d\omega \quad \text{1D Continuous}$$

$$\omega(\vec{k}) \Downarrow$$

$$\iiint E_0(\vec{k})\, e^{i[\vec{k}\cdot\vec{r} - \omega t]}\, dk_x dk_y dk_z \quad \text{3D}$$

Irradiance :

$$|E(\vec{r},t)|^2 = E^*(\vec{r},t)\, E(\vec{r},t)$$

: $\propto$ average energy density
average Intensity.

7.1 Addition of Waves of the same frequency

$$E(z,t) = E_0 e^{i(kz-\omega t+\varphi)}$$

$$|E(z,t)|^2 = E_0^* e^{-i(kz-\omega t+\varphi)} E_0 e^{i(kz-\omega t+\phi)} = E_0^* E_0 = |E_0|^2$$

1D :

$$E_1 = E_{10}\, e^{i(k_1 z - \omega t + \varphi_1)}$$

$$E_2 = E_{20}\, e^{i[k_2 z - \omega t + \varphi_2)}$$

$$E = E_1 + E_2 = E_{10}\, e^{i(k_1 z - \omega t + \varphi_1)} + E_{20}\, e^{i(k_2 z - \omega t + \varphi_2)}$$

* for $k_1 = k_2 = k$, moving in the same direction (also let $E_{10}, E_{20} \in R$)

$$E = E_{10} e^{i(kz-\omega t+\varphi_1)} + E_{20} e^{i(kz-\omega t+\varphi_2)} = E_1 + E_2$$

$$|E|^2 = (E_1^* + E_2^*)(E_1+E_2) = E_1^*E_1 + E_2^*E_2 + E_1^*E_2 + E_2^*E_1$$

$$= E_{10}^2 + E_{20}^2 + E_{10}E_{20} e^{i(\varphi_2-\varphi_1)} + E_{10}E_{20} e^{-i(\varphi_2-\varphi_1)}$$

$$= E_{10}^2 + E_{20}^2 + 2E_{10}E_{20}\cos(\varphi_2-\varphi_1)$$

1) $\varphi_2 - \varphi_1 = 0$

$$|E|^2 = E_{10}^2 + E_{20}^2 + 2E_{10}E_{20} = |E_{10}+E_{20}|^2$$

2) $\varphi_2 - \varphi_1 = \pi$

$$|E|^2 = E_{10}^2 + E_{20}^2 - 2E_{10}E_{20} = (E_{10}-E_{20})^2$$

* for $k_1 = k$, $k_2 = -k$, moving in opposite directions. ($E_{10}, E_{20} \in R$)

$$E = E_{10} e^{i(kz-\omega t+\varphi_1)} + E_{20} e^{i(-kz-\omega t+\varphi_2)} = E_1 + E_2$$

$$|E|^2 = E_{10}^2 + E_{20}^2 + E_{10}E_{20} e^{i(-2kz+\varphi_2-\varphi_1)} + E_{10}E_{20} e^{-i(-2kz+\varphi_2-\varphi_1)}$$

$$= E_{10}^2 + E_{20}^2 + 2E_{10}E_{20}\cos(\varphi_2-\varphi_1-2kz)$$

1) Standing wave $\quad E_{10} = E_{20}$

$$|E|^2 = 2E_{10}^2 + 2E_{10}^2\cos(\varphi_2-\varphi_1-2kz) = 2E_{10}^2[1+\cos(\varphi_2-\varphi_1-2kz)]$$

$$= 4E_{10}^2\cos^2\left(\frac{\varphi_2-\varphi_1}{2} - kz\right)$$

$$E = E_{10}\, e^{i(kz-\omega t+\varphi_1)} + E_{10}\, e^{i(-kz-\omega t+\varphi_2)}$$

$$= E_{10}\left[e^{i(kz+\varphi_1)} + e^{-i(kz-\varphi_2)}\right] e^{-i\omega t}$$

$$= E_{10}\left[e^{i\varphi_1} e^{ikz} + e^{i\varphi_2} e^{-ikz}\right] e^{-i\omega t}$$

$$\xrightarrow[\varphi_2-\varphi_1=\Delta\varphi \qquad \varphi_2=\frac{1}{2}(\varphi+\Delta\varphi)]{\varphi_1+\varphi_2=\varphi \qquad \varphi_1=\frac{1}{2}(\varphi-\Delta\varphi)} E_{10}\left[e^{\frac{i}{2}(\varphi-\Delta\varphi)} e^{ikz} + e^{\frac{i}{2}(\varphi+\Delta\varphi)} e^{-ikz}\right] e^{-i\omega t}$$

$$= E_{10}\, e^{\frac{i}{2}\varphi}\left[e^{i(kz-\frac{\Delta\varphi}{2})} + e^{-i(kz-\frac{\Delta\varphi}{2})}\right] e^{-i\omega t}$$

$$= 2E_{10}\, e^{\frac{i}{2}\varphi} \cos\left(kz-\frac{\Delta\varphi}{2}\right) e^{-i\omega t}$$

$$\Rightarrow |E|^2 = 4E_{10}^2 \cos^2\left(kz-\frac{\Delta\varphi}{2}\right)$$

7.2. Addition of Waves of Different Frequency

$$E_1 = E_{01}\, e^{i(k_1 z-\omega_1 t+\varphi_1)} \qquad E_{01},\ E_{02} \in R.$$

$$E_2 = E_{02}\, e^{i(k_2 z-\omega_2 t+\varphi_2)}$$

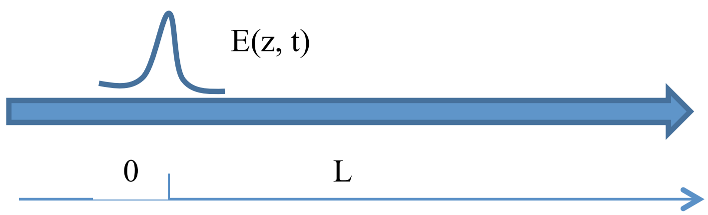
$$|E|^2 = |E_1+E_2|^2 = E_{10}^2 + E_{20}^2 + E_{01}E_{02}\, e^{i[(k_2-k_1)z-(\omega_2-\omega_1)t+\varphi_2-\varphi_1]}$$
$$+ E_{01}E_{02}\, e^{-i[(k_2-k_1)z-(\omega_2-\omega_1)t+\varphi_2-\varphi_1]}$$

$$= E_{10}^2 + E_{20}^2 + \underbrace{2E_{01}E_{02}\cos\left[(k_2-k_1)z-(\omega_2-\omega_1)t+\varphi_2-\varphi_1\right]}_{\text{Beat}}$$

beat frequency $|\omega_2-\omega_1|$

PHYS3038 Lecture Note 10

Group Velocity and Pulse propagation (confinement in time)



Here we take plane wave for example. Laser output has frequency ω_0. At z=0, modulate the laser output and the pulse can described as

$$E(z=0,t) = \frac{1}{\sqrt{2\pi}}\int F(\omega)e^{-i\omega t}d\omega \quad (27)$$

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int E(0,t)e^{i\omega t}d\omega \quad (28)$$

For each frequency component ω, we know it propagates as $e^{i[k(\omega)z-\omega t]}$. For the pulse propagation, we extend (27) to (wave packet):

$$E(z,t) = \frac{1}{\sqrt{2\pi}}\int F(\omega)e^{i[k(\omega)z-\omega t]}d\omega \quad (29)$$

As we know, the wave number k is frequency dependent. From the Fourier transforms (27) and (28), for pulse with finite time duration, the frequency spectrum has also a finite width with center at ω_0. Using Taylor expansion we can approximate the wave number as

$$k(\omega) = k(\omega_0) + (\omega-\omega_0)\frac{dk}{d\omega}|_{\omega_0} + \cdots \cong k(\omega_0) + (\omega-\omega_0)\frac{dk}{d\omega}|_{\omega_0}$$
$$= k(\omega_0) + (\omega-\omega_0)\frac{dk}{d\omega}|_{\omega_0} + \cdots \cong k(\omega_0) + (\omega-\omega_0)\frac{1}{V_g(\omega_0)} \quad (30)$$

Where we define $\frac{1}{V_g(\omega_0)} = \frac{dk}{d\omega}|_{\omega_0}$ or $V_g(\omega_0) = \frac{d\omega}{dk}|_{\omega_0}$. (31)

Then we rewrite (29) to

$$E(z,t) = \frac{1}{\sqrt{2\pi}}\int F(\omega)e^{i\left[k(\omega_0)z+(\omega-\omega_0)\frac{z}{V_g(\omega_0)}-\omega t\right]}d\omega$$
$$= \frac{1}{\sqrt{2\pi}}\int F(\omega)e^{i\left[k(\omega_0)z+\frac{\omega z}{V_g(\omega_0)}-\frac{\omega_0 z}{V_g(\omega_0)}-\omega t\right]}d\omega$$
$$= \frac{1}{\sqrt{2\pi}}\int F(\omega)e^{i\left[k(\omega_0)z-\frac{\omega_0 z}{V_g(\omega_0)}-\omega t+\frac{\omega z}{V_g(\omega_0)}\right]}d\omega$$
$$= \frac{1}{\sqrt{2\pi}}e^{i\left[k(\omega_0)-\frac{\omega_0}{V_g(\omega_0)}\right]z}\int F(\omega)e^{-i\omega\left[t-\frac{z}{V_g(\omega_0)}\right]}d\omega$$
$$= e^{i\left[k(\omega_0)-\frac{\omega_0}{V_g(\omega_0)}\right]z}E(0,t-\frac{z}{V_g(\omega_0)})$$

$$E(z,t) = e^{i\left[k(\omega_0) - \frac{\omega_0}{V_g(\omega_0)}\right]z} E(0, t - \frac{z}{V_g(\omega_0)}) \tag{32}$$

As you see, at z, the E field amplitude reassembles the shape at z=0 but with a time delay $\frac{z}{V_g(\omega_0)}$. We define $V_g(\omega_0)$ as the group velocity that describes the motion of the center of the entire wave packet.

Example: Vacuum

$$k(\omega) = \frac{\omega}{c}$$
$$k(\omega_0) = \frac{\omega_0}{c}$$
$$\frac{1}{V_g} = \frac{dk}{d\omega} = \frac{1}{c} \Rightarrow V_g = c$$
$$\frac{\omega_0}{V_g(\omega_0)} = \frac{\omega_0}{c} = k(\omega_0)$$

From (32) we have for the pulse propagation in vacuum:

$$E(z,t) = E(0, t - \frac{z}{c})$$

which travels with a speed of c.