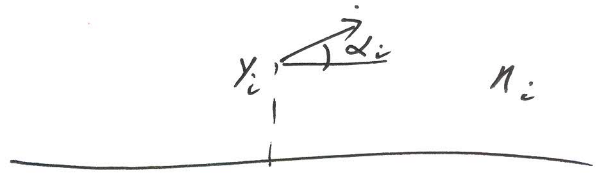
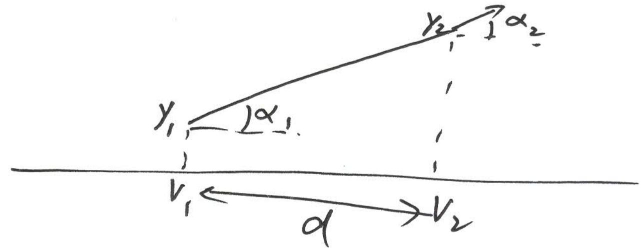
6.2. Analytical Ray Tracing — Matrix Method

(1) Ray Vector $\quad \vec{r}_i = \begin{pmatrix} n_i \alpha_i \\ y_i \end{pmatrix}$



(2) Free Space



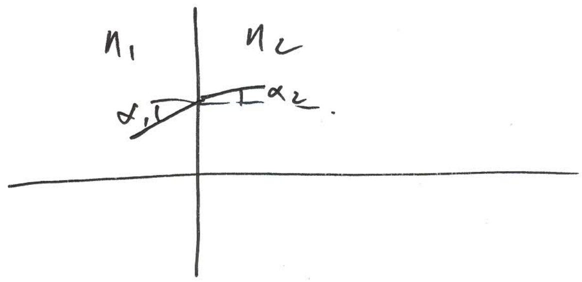
$$\alpha_2 = \alpha_1 \Leftrightarrow n\alpha_2 = n\alpha_1$$

$$y_2 = y_1 + d\tan\alpha_1 = y_1 + d\alpha_1 = y_1 + \frac{d}{n} n\alpha_1 = \frac{d}{n} n\alpha_1 + y_1$$

$$\Rightarrow \begin{bmatrix} n\alpha_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{d}{n} & 1 \end{bmatrix} \begin{bmatrix} n\alpha_1 \\ y_1 \end{bmatrix} \Leftrightarrow \vec{r}_2 = T_{21}\vec{r}_1$$

Transfer Matrix $\quad T_{21} = \begin{bmatrix} 1 & 0 \\ \frac{d}{n} & 1 \end{bmatrix}$

(3) Refraction



$$n_1 \sin\alpha_1 = n_2 \sin\alpha_2$$

$$n_1 \alpha_1 = n_2 \alpha_2$$

$$n_2 \alpha_2 = n_1 \alpha_1$$

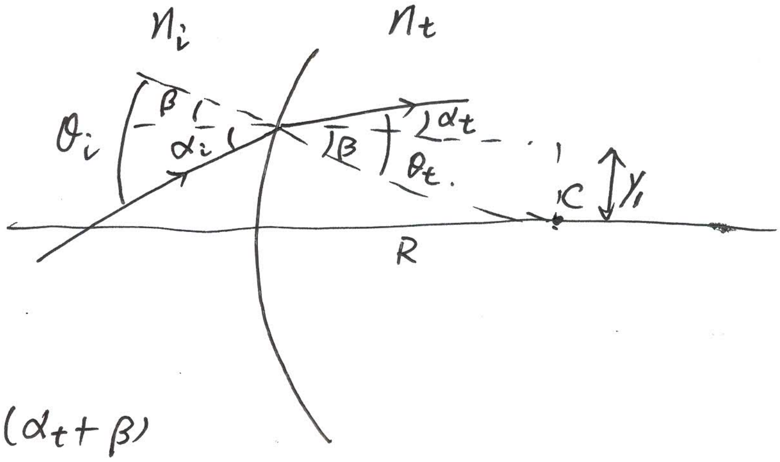
$$y_2 = y_1$$

$$\Rightarrow \quad \vec{r}_2 = \vec{r}_1 = I\vec{r}_1 \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(4) propagation from n_1 to n_2

$$T = \begin{bmatrix} 1 & 0 \\ \frac{d_2}{n_2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{d_1}{n_1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{d_1}{n_1} + \frac{d_2}{n_2} & 1 \end{bmatrix}$$

[5] Refraction Matrix



$$n_i\theta_i = n_t\theta_t$$

$$n_i(\alpha_i+\beta) = n_t(\alpha_t+\beta)$$

$$\beta = \frac{y_1}{R}$$

$$\Rightarrow n_i\left(\alpha_i+\frac{y_1}{R}\right) = n_t\left(\alpha_t+\frac{y_1}{R}\right)$$

$$\Rightarrow n_i\alpha_i + \frac{n_i}{R}y_1 = n_t\alpha_t + \frac{n_t}{R}y_1$$

$$\Rightarrow n_t\alpha_t = n_i\alpha_i + \frac{(n_i-n_t)}{R}y_1 = n_i\alpha_i - \left(\frac{n_t-n_i}{R}\right)y_1$$

$$y_2 = y_1$$

$$\Rightarrow \begin{pmatrix} n_t\alpha_t \\ y_2 \end{pmatrix} = \begin{bmatrix} 1 & -\left(\frac{n_t-n_i}{R}\right) \\ 0 & 1 \end{bmatrix}\begin{bmatrix} n_i\alpha_i \\ y_1 \end{bmatrix}$$

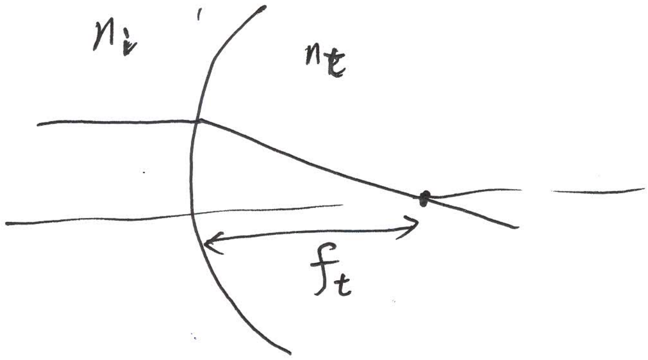
Refraction Matrix

$$R = \begin{bmatrix} 1 & -\left(\frac{n_t-n_i}{R}\right) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -D \\ 0 & 1 \end{bmatrix}$$

power of Refration surfae $D = \dfrac{n_t-n_i}{R}$

For flat surfae $R \Rightarrow \infty$, $D \rightarrow 0$ $R \rightarrow I$

61 Refration + propagation



$$H = TR = \begin{bmatrix} 1 & 0 \\ \frac{d}{n_t} & 1 \end{bmatrix} \begin{bmatrix} 1 & -D \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -D \\ \frac{d}{n_t} & 1-\frac{Dd}{n_t} \end{bmatrix}$$

Let $y_1 = y_1$

$\alpha_1 = 0$

$$\begin{bmatrix} n_t \alpha_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -D \\ \frac{d}{n_t} & 1-\frac{Dd}{n_t} \end{bmatrix} \begin{bmatrix} 0 \\ y_1 \end{bmatrix} = \begin{bmatrix} -D y_1 \\ \left(1-\frac{Dd}{n_t}\right) y_1 \end{bmatrix} = \begin{bmatrix} -D \\ 1-\frac{Dd}{n_t} \end{bmatrix} y_1$$

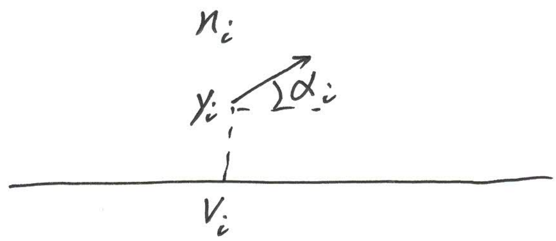
for $y_2 = 0$ at the focal point:

$$1 - \frac{Dd}{n_t} = 0 \Rightarrow f = d = \frac{n_t}{D} = \frac{n_t}{n_t - n_i} R.$$

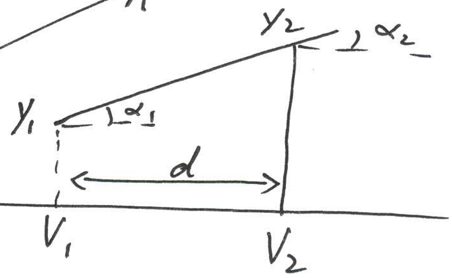
Rewrite $R = \begin{bmatrix} 1 & -D \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{n_t}{f} \\ 0 & 1 \end{bmatrix}$

6.2. Analytical Ray Tracing – ABCD Matrix Method

(1) Ray Vector $\vec{r}_i = \begin{pmatrix} n_i\alpha_i \\ y_i \end{pmatrix}$



(2) Free Space



$$\alpha_2 = \alpha_1$$

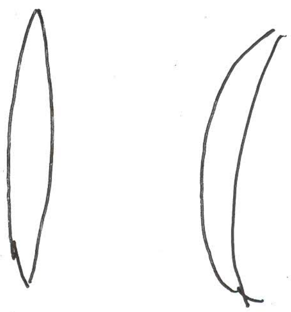
$$y_2 = y_1 + d\tan\alpha_1$$
$$\approx y_1 + d\alpha_1$$

$\alpha \approx \sin\alpha \approx \tan\alpha$

$$\Rightarrow n y_2 = n y_1 + n d\alpha_1$$

$$\Rightarrow \begin{bmatrix} n\alpha_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & nd \end{bmatrix} \begin{bmatrix} n\alpha_1 \\ y_1 \end{bmatrix} \Longleftrightarrow \vec{r}_2 =$$

(3) Thin Lens



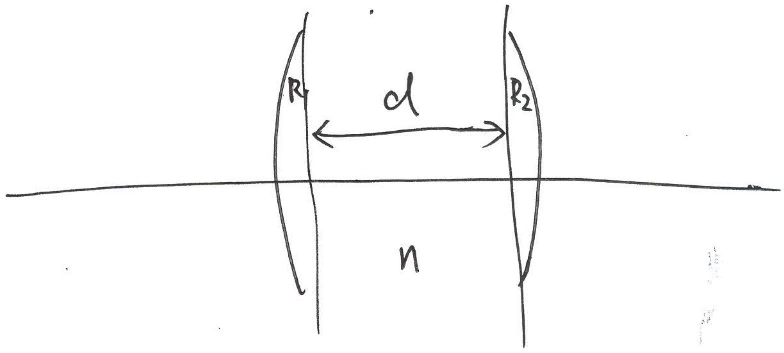
$$L = R_2 R_1$$
$$= \begin{bmatrix} 1 & -D_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -D_1 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -(D_1+D_2) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1/f \\ 0 & 1 \end{bmatrix}$$

Let $\alpha_1 = 0$, in air.

$$\begin{bmatrix} \alpha_2 \\ y_2 \end{bmatrix} = T(d) L \begin{bmatrix} 0 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} \begin{bmatrix} 1 & -1/f \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 & -1/f \\ d & 1-\frac{d}{f} \end{bmatrix} \begin{bmatrix} 0 \\ y_1 \end{bmatrix} = \begin{bmatrix} -1/f \\ 1-\frac{d}{f} \end{bmatrix} y_1$$

at focal point $y_2 = 0 \Rightarrow \boxed{d = f}$

(8) Thick lens.



$$L = R_2\, T\!\left(\frac{d}{n}\right) R_1$$

$$= \begin{bmatrix} 1 & -D_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{d}{n} & 1 \end{bmatrix} \begin{bmatrix} 1 & -D_1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{D_2 d}{n} & -D_2 \\ \frac{d}{n} & 1 \end{bmatrix} \begin{bmatrix} 1 & -D_1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{D_2 d}{n} & -D_1 - D_2 - \frac{D_1 D_2 d}{n} \\ \frac{d}{n} & \frac{-D_1 d}{n} + 1 \end{bmatrix}$$