1. To show that

$$E_z = \frac{\sigma R^2}{4\pi\varepsilon_0}\int_S \frac{z - R\cos\theta}{\left(R^2 + z^2 - 2Rz\cos\theta\right)^{3/2}} \sin\theta d\theta d\phi = \begin{cases} \dfrac{q}{4\pi\varepsilon_0 z^2} & \text{for } z > R \\ 0 & \text{for } z < R \end{cases}$$

Solution:

$$E_z = \frac{\sigma R^2}{4\pi\varepsilon_0}\int_0^{2\pi} d\phi \int_0^{\pi} \frac{\left(z - R\cos\theta\right)\sin\theta d\theta}{\left(R^2 + z^2 - 2Rz\cos\theta\right)^{3/2}}$$

$$= \frac{\sigma R^2}{4\pi\varepsilon_0} 2\pi \int_\pi^0 \frac{\left(z - R\cos\theta\right) d\left(\cos\theta\right)}{\left(R^2 + z^2 - 2Rz\cos\theta\right)^{3/2}}$$

$$= \frac{\sigma R^2}{2\varepsilon_0}\int_\pi^0 \frac{\left(z - R\cos\theta\right) d\left(\cos\theta\right)}{\left(R^2 + z^2 - 2Rz\cos\theta\right)^{3/2}}$$

Let $\alpha = R^2 + z^2 - 2Rz\cos\theta$

$$E_z = \frac{\sigma R^2}{2\varepsilon_0}\int_{(z+R)^2}^{(z-R)^2} \frac{\alpha + z^2 - R^2}{2z} \frac{1}{-2Rz} \frac{d\alpha}{\alpha^{3/2}}$$

$$= \frac{\sigma R}{8\varepsilon_0 z^2}\int_{(z+R)^2}^{(z-R)^2} \left(\frac{R^2 - z^2}{\alpha^{3/2}} - \frac{1}{\alpha^{1/2}}\right) d\alpha$$

$$= \frac{\sigma R}{8\varepsilon_0 z^2}\left[\left(R^2 - z^2\right)\frac{\alpha^{-1/2}}{-1/2} - \frac{\alpha^{1/2}}{1/2}\right]_{(z+R)^2}^{(z-R)^2}$$

$$= \frac{\sigma R}{4\varepsilon_0 z^2}\left[\left(z^2 - R^2\right)\frac{1}{\alpha^{1/2}} - \alpha^{1/2}\right]_{(z+R)^2}^{(z-R)^2}$$

For $z > R$,

$$E_z = \frac{\sigma R}{4\varepsilon_0 z^2}\left[\left(\left(z^2 - R^2\right)\frac{1}{z - R} - \left(z - R\right)\right) - \left(\left(z^2 - R^2\right)\frac{1}{z + R} - \left(z + R\right)\right)\right]$$

$$= \frac{\sigma R}{4\varepsilon_0 z^2}\left[\left(\left(z + R\right) - \left(z - R\right)\right) - \left(\left(z - R\right) - \left(z + R\right)\right)\right]$$

$$= \frac{\sigma R}{4\varepsilon_0 z^2}\left[2R + 2R\right]$$

$$= \frac{\sigma R^2}{\varepsilon_0 z^2}$$

$$= \frac{\sigma \times 4\pi R^2}{4\pi\varepsilon_0 z^2}$$

$$= \frac{q}{4\pi\varepsilon_0 z^2}$$

For $z < R$,

$$E_z = \frac{\sigma R}{4\varepsilon_0 z^2}\left[\left(\left(z^2 - R^2\right)\frac{1}{R-z} - \left(R-z\right)\right) - \left(\left(z^2 - R^2\right)\frac{1}{z+R} - \left(z+R\right)\right)\right]$$

$$= \frac{\sigma R}{4\varepsilon_0 z^2}\left[\left(-\left(z+R\right) - \left(R-z\right)\right) - \left(\left(z-R\right) - \left(z+R\right)\right)\right]$$

$$= \frac{\sigma R}{4\varepsilon_0 z^2}\left[-2R + 2R\right]$$

$$= 0$$

2. To show that

$$\int_0^\pi \frac{\sin\theta}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}}d\theta = \begin{cases} \dfrac{2}{r} & \text{for } r > r' \\ \dfrac{2}{r'} & \text{for } r' > r \end{cases}$$

Solution:

$$\int_0^\pi \frac{\sin\theta}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}}d\theta = -\int_0^\pi \frac{d\left(\cos\theta\right)}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}}$$

$$= \frac{1}{2rr'}\int_0^\pi \frac{d\left(r^2 + r'^2 - 2rr'\cos\theta\right)}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}}$$

$$= \left[\frac{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}}{rr'}\right]_0^\pi$$

$$= \frac{r + r' - \sqrt{\left(r - r'\right)^2}}{rr'}$$

For $r > r'$

$$\int_0^\pi \frac{\sin\theta}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}}d\theta = \frac{r + r' - \left(r - r'\right)}{rr'} = \frac{2}{r}$$

For $r < r'$

$$\int_0^\pi \frac{\sin\theta}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}}d\theta = \frac{r + r' - \left(r' - r\right)}{rr'} = \frac{2}{r'}$$