Name: _______________ Student ID: __________

PHYS 3033 Electricity and Magnetism I

Quiz 3

22 September 2015

Time allowed 20 minutes

a) Find the electric field due to the following volume charge distribution:

$$\rho(\mathbf{r}') = \begin{cases} \rho_0 r'/R & \text{for } 0 \le r' \le R \\ 0 & \text{for } r' > R \end{cases}$$

Where $r' = |\mathbf{r}'|$ and ρ_0 and R are constants.

b) Using the result of a), find the electric potential for $0 \le r' \le R$ and $r' > R$.

Solution

a)

Due to spherical symmetry, the E field must be of the form $\mathbf{E}(\mathbf{r}) = E(r)\hat{\mathbf{r}}$.

By Gauss's law, with a Gaussian surface centered at the origin and with radius r, we have

$$E(r)\times 4\pi r^2 = \frac{q_{\text{enc}}}{\varepsilon_0} \Rightarrow E(r) = \frac{q_{\text{enc}}}{4\pi\varepsilon_0 r^2}.$$

For $r > R$,

$$q_{\text{enc}} = \int_0^R \rho(r')4\pi r'^2 dr' = \int_0^R \frac{\rho_0 r'}{R} 4\pi r'^2 dr' = \frac{4\pi\rho_0}{R}\int_0^R r'^3 dr' = \frac{4\pi\rho_0}{R}\frac{R^4}{4} = \pi\rho_0 R^3.$$

Hence
$$E(r) = \frac{\pi\rho_0 R^3}{4\pi\varepsilon_0 r^2} = \frac{\rho_0 R^3}{4\varepsilon_0 r^2}.$$

For $0 \le r \le R$,

$$q_{\text{enc}} = \int_0^r \rho(r')4\pi r'^2 dr' = \int_0^r \frac{\rho_0 r'}{R} 4\pi r'^2 dr' = \frac{4\pi\rho_0}{R}\int_0^r r'^3 dr' = \frac{4\pi\rho_0}{R}\frac{r^4}{4} = \frac{\pi\rho_0 r^4}{R}.$$

Hence
$$E(r) = \frac{\pi\rho_0 r^4 / R}{4\pi\varepsilon_0 r^2} = \frac{\rho_0 r^2}{4\varepsilon_0 R}.$$

In conclusion,

$$\mathbf{E}(\mathbf{r}) = \begin{cases} \dfrac{\rho_0 r^2}{4\varepsilon_0 R}\hat{\mathbf{r}} & \text{for } 0 \le r \le R \\ \dfrac{\rho_0 R^3}{4\varepsilon_0 r^2}\hat{\mathbf{r}} & \text{for } r > R \end{cases}$$

b)

The electric potential V($\vec{r}$) ($\vec{r} = (r, \theta, \varphi)$) is defined as V($\vec{r}$) =$-\int_\infty^{\vec{r}} \vec{E}\cdot d\vec{l}$, if we set the zero potential point to be infinity.

By spherical symmetry, V($\vec{r}$) only depends on r, i.e. $V(r, \theta, \varphi) = V(r)$, and hence

$$V(r) = -\int_\infty^r E(r')\,dr'\text{, where r' is only a dummy variable.}$$

For $r \ge R$,

$$V(r) = -\int_\infty^r \frac{\rho_0 R^3}{4\varepsilon_0 r'^2} dr' = \left(\frac{\rho_0 R^3}{4\varepsilon_0}\right)\left(-\int_\infty^r \frac{1}{r'^2} dr'\right) = \left(\frac{\rho_0 R^3}{4\varepsilon_0}\right)\left(\frac{1}{r'}\right)\Big|_\infty^r = \left(\frac{\rho_0 R^3}{4\varepsilon_0}\right)\left(\frac{1}{r}\right)$$

For $r < R$,

$$V(r) = -\int_{\infty}^{r} E(r')\,dr' = -\left(\int_{\infty}^{R} E(r')\,dr' + \int_{R}^{r} E(r')\,dr'\right)$$

$$= -\left(\int_{\infty}^{R} \frac{\rho_0 R^3}{4\varepsilon_0 r'^2} dr' + \int_{R}^{r} \frac{\rho_0 r'^2}{4\varepsilon_0 R} dr'\right) = \left(\frac{\rho_0 R^3}{4\varepsilon_0}\right)\left(\frac{1}{R}\right) + \left(\frac{\rho_0}{4\varepsilon_0 R}\right)\left(\int_{r}^{R} r'^2\,dr'\right)$$

$$= \left(\frac{\rho_0 R^2}{4\varepsilon_0}\right) + \left(\frac{\rho_0}{12\varepsilon_0 R}\right)(R^3 - r^3)$$

Note that $V(r \to R^-) = \frac{\rho_0 R^2}{4\varepsilon_0} = V(r \to R^+)$, and that $E(r) = -\frac{dV(r)}{dr}$.