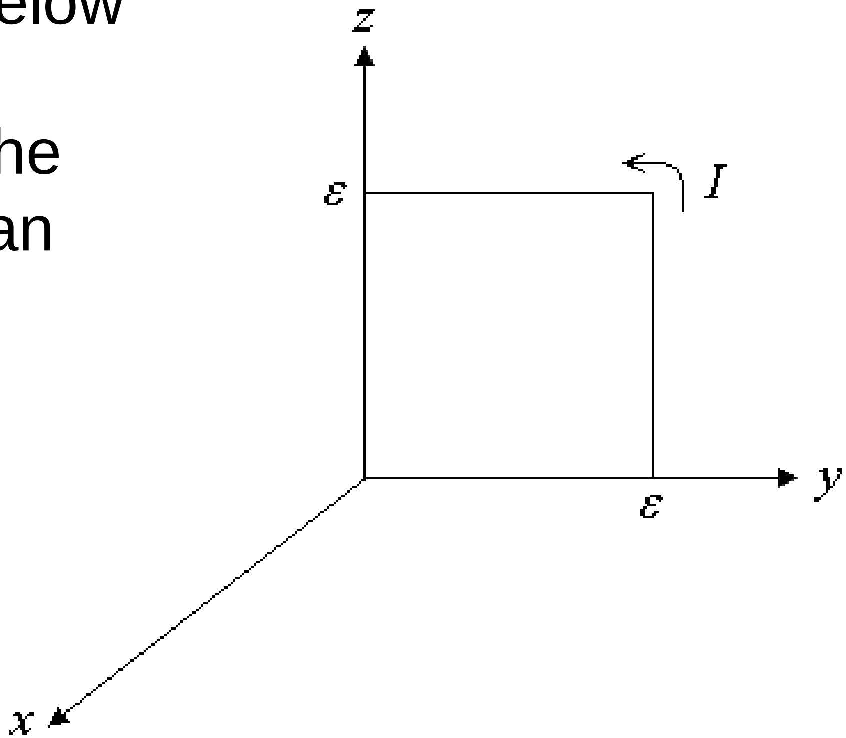
Force on an ideal magnetic dipole with dipole moment $\mathbf{m}$ in a B-field $\mathbf{B}$ is given by

$$\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})$$

A simple proof of this is to consider a square current loop with side-length ε, as shown below



By considering the forces on the four sides of the square, we can prove that, when $\varepsilon \to 0$

$$\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})$$

(Proof in assignment)

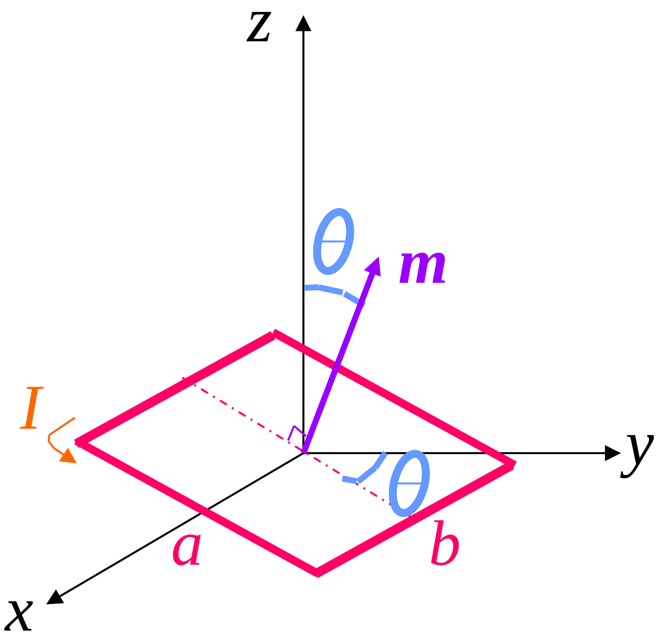
The torque on $\mathbf{N}$ in a uniform magnetic field $\mathbf{B}$ is given by

$$\mathbf{N} = \mathbf{m} \times \mathbf{B}$$

Proof:

Consider a rectangular loop with sides a, b and making an angle θ with $\mathbf{B}$.

Without loss of generality, let $\mathbf{B}$ point along the *z*-direction and the loop is tilted from the *z*-axis towards the *y*-axis by an angle θ.

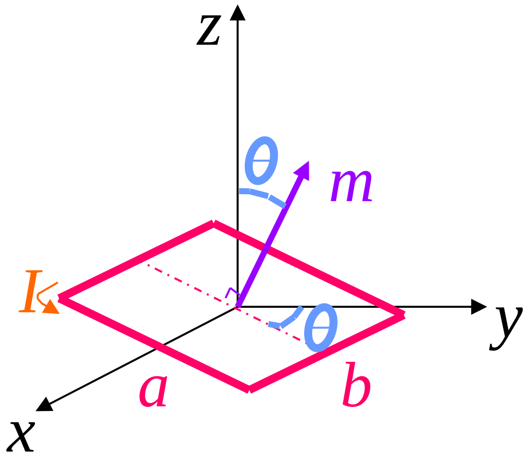


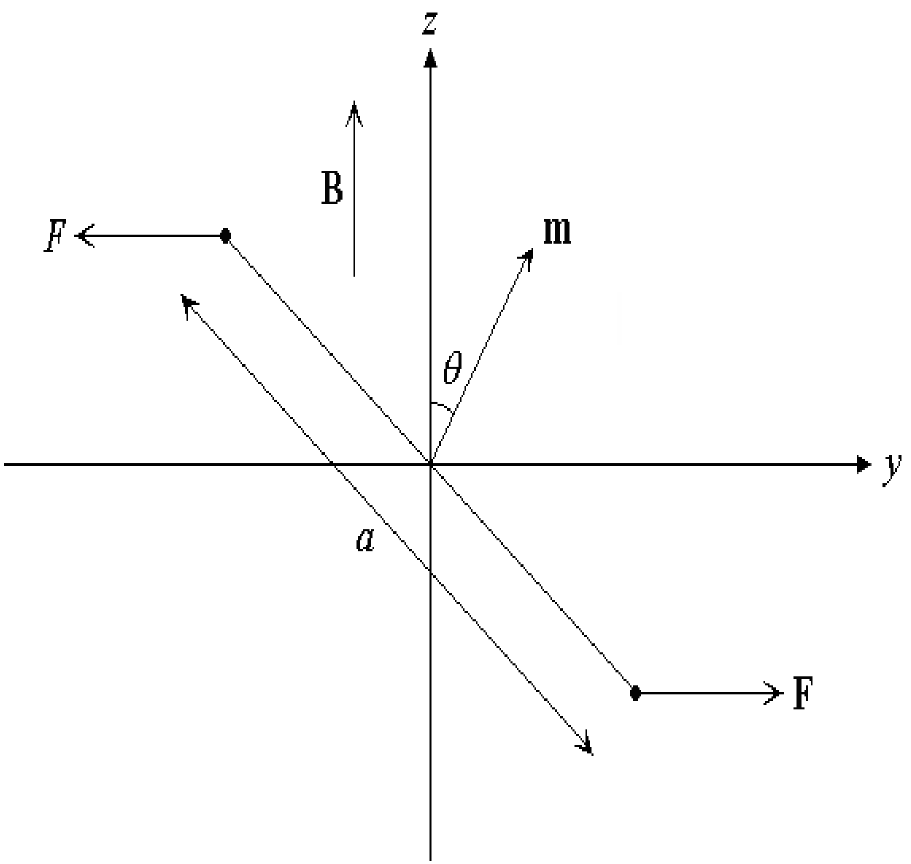
$$F = IbB$$

$$\mathbf{N} = aF\sin\theta\,\hat{\mathbf{x}}$$

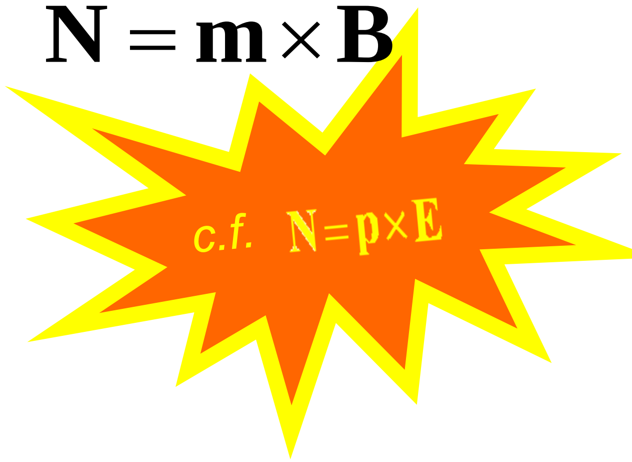
$$= IabB\sin\theta\,\hat{\mathbf{x}}$$

$$= mB\sin\theta\,\hat{\mathbf{x}}$$





In vector form: $\mathbf{N} = \mathbf{m} \times \mathbf{B}$



Magnetic Field in Matters

Paramagnetism

What is Paramagnetism?

Inside matters, there are a lot of ***tiny currents*** due to the electrons orbiting around the nuclei and intrinsic spins.

The scale of these small ***“current loops”*** are so small that the applied B-field can be considered uniform.

In a ***uniform*** B-field, $\mathbf{m}\cdot\mathbf{B}$ is ***constant*** and so

$$\mathbf{F}=\nabla\left(\mathbf{m}\cdot\mathbf{B}\right)=0$$

The ***torque***, $\mathbf{N}=\mathbf{m}\times\mathbf{B}$, however, is ***non-zero*** & ***tends to align*** $\mathbf{m}$ ***in the same direction of*** $\mathbf{B}$

This effect is called ***paramagnetism***.

Magnetic Field in Matters

Diamagnetism

What is Diamagnetism?

$$\Delta \mathbf{m} \propto -\mathbf{B}$$

The induced net magnetic dipole moment is ***opposite*** to the field.

This effect is called ***diamagnetism***.

Effect of Magnetic field on Atomic Orbits

For electrons orbiting around nuclei, in addition to the paramagnetic effect, there is another less significant effect due to an external B-field.

A full, rigorous treatment of this effect requires quantum mechanics. Here is a qualitative analysis:

Let the speed of an orbiting electron be *v*, then the period

$$T = \frac{2\pi R}{v}$$

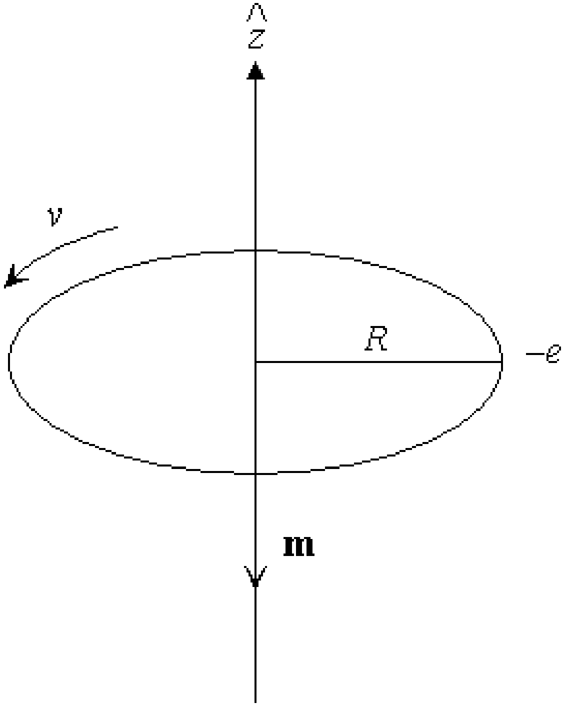
where *R* is the radius of the orbit.

This yields a steady current

$$I = \frac{e}{T} = \frac{ev}{2\pi R}$$

The orbital dipole moment is therefore

$$\begin{aligned} \mathbf{m} &= I\mathbf{a} \\ &= -I\pi R^2\ \hat{\mathbf{z}} \\ &= -\frac{1}{2}evR\ \hat{\mathbf{z}} \end{aligned}$$

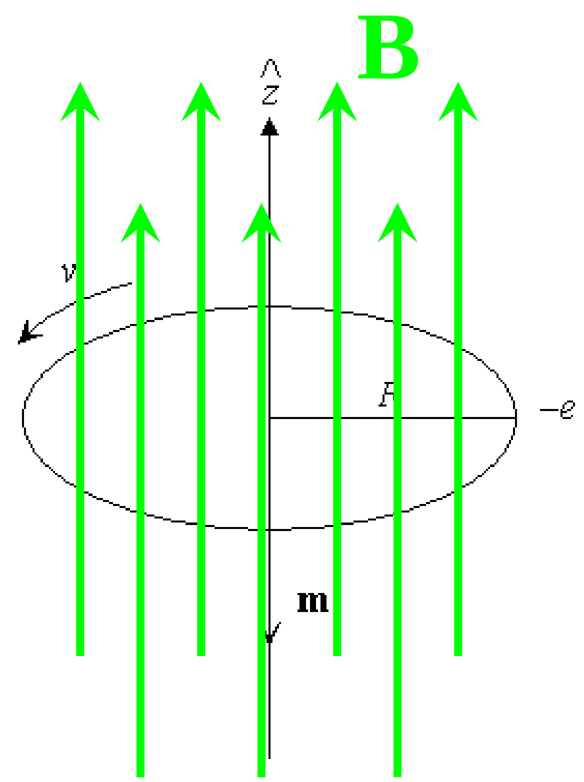


Suppose now a B-field $\mathbf{B} = B\hat{\mathbf{z}}$ is applied.

Then, in addition to the electric force between the electron and the nucleus, there is a magnetic force.

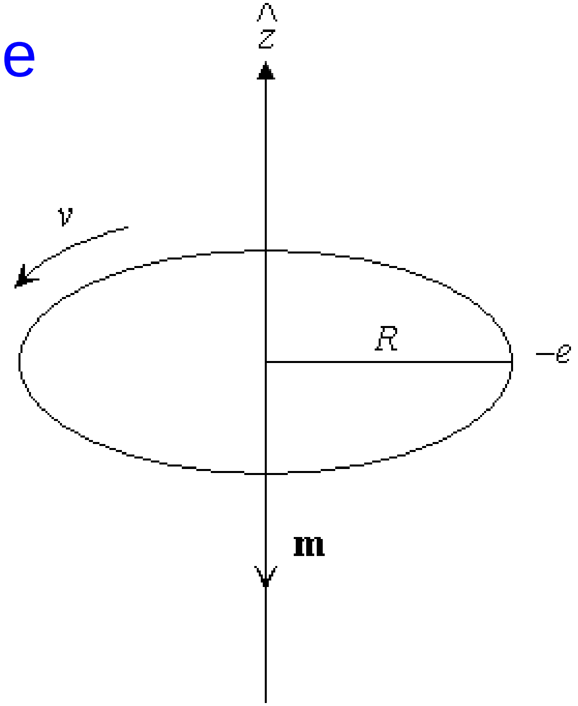
In the absence of the B-field, the centripetal force is contributed by the electric force alone, so

$$\frac{1}{4\pi\varepsilon_0}\frac{e^2}{R^2} = m_e\frac{v^2}{R}$$



With the magnetic field, the velocity of the electron is changed from *v* to $\bar{v}$, so that

$$\frac{1}{4\pi\varepsilon_0}\frac{e^2}{R^2} + e\bar{v}B = m_e\frac{\bar{v}^2}{R}$$



Let $\bar{v} - v = \Delta v$

$$\therefore \quad m_e\frac{v^2}{R} + eB\bar{v} = m_e\frac{\bar{v}^2}{R}$$

$$\frac{m_e}{R}\left(\bar{v} - \Delta v\right)^2 + eB\bar{v} = m_e\frac{\bar{v}^2}{R}$$

$$eB\bar{v} - 2\frac{m_e\bar{v}}{R}\Delta v + \frac{m_e}{R}(\Delta v)^2 = 0$$

$$eB\bar{v} - 2\frac{m_e \bar{v}}{R}\Delta v + \frac{m_e}{R}(\Delta v)^2 = 0$$

Assume that $\frac{\Delta v}{\bar{v}}$ is so small that the second-order term can be ignored, then $\Delta v = \frac{eRB}{2m_e}$

Therefore, the electron speeds up or slows down, depending on the direction of the applied field.

When $B > 0$, i.e. $\mathbf{B}$ is along the z-direction, it speeds up.

When $B < 0$, i.e. $\mathbf{B}$ is along the -ve z-direction, it slows down.

$$\because \quad \mathbf{m} = -\frac{1}{2}evR\,\hat{\mathbf{z}}$$

$$\Delta\mathbf{m} = -\frac{1}{2}e(\Delta v)R\,\hat{\mathbf{z}}$$

$$= -\frac{e^2R^2}{4m_e}\mathbf{B}$$

Magnetic Field in Matters

Ferromagnetism

What is Ferromagnetism?

- “Forzen in” magnetic dipoles
- Not due to external fields, but sustained by interaction between nearby dipoles
- Quantum mechanics → Dipoles “like” to point in the same direction as their neighbors.
- Emphatically non-linear

Magnetic Field in Matters

Magnetization

What is Magnetization?

The magnetization of an object, $\mathbf{M}$, is defined as the amount of magnetic dipole moment per unit volume

$$\mathbf{M} \equiv \text{Magnetic dipole moment per unit volume.}$$

Magnetic Field in Matters

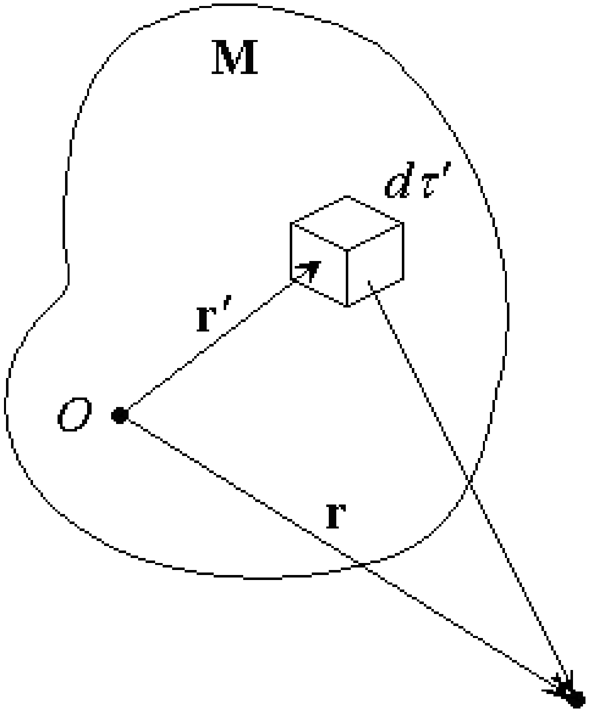
The Field of a Magnetized Object

The vector potential $\mathbf{A}$ of a single ideal magnetic dipole moment $\mathbf{m}$ is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

So, for a magnetized object with magnetization $\mathbf{M}$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{M}(\mathbf{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2} d\tau'$$



Similar to the case in electric polarization, the potential can be re-written in another form which gives better physical interpretation.

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{M}(\mathbf{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'$$

Recall: $\nabla'\frac{1}{\mathscr{r}} = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$

$$\therefore \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \mathbf{M}(\mathbf{r}')\times(\nabla'\frac{1}{\mathscr{r}})\,d\tau'$$

$$\because \quad \nabla'\times\left(\frac{1}{\mathscr{r}}\mathbf{M}(\mathbf{r}')\right) = \frac{1}{\mathscr{r}}\nabla'\times\mathbf{M}(\mathbf{r}') - \mathbf{M}(\mathbf{r}')\times(\nabla'\frac{1}{\mathscr{r}})$$

$$\therefore \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\left[\frac{1}{\mathscr{r}}\nabla'\times\mathbf{M}(\mathbf{r}') - \nabla'\times\left(\frac{1}{\mathscr{r}}\mathbf{M}(\mathbf{r}')\right)\right]d\tau'$$

Consider the second term in the integral. We shall proof later that

$$\int \nabla' \times \left(\frac{1}{r} \mathbf{M}(\mathbf{r}') \right) d\tau' = -\oint_S \frac{1}{r} \mathbf{M}(\mathbf{r}') \times d\mathbf{a}'$$

where *S* is the surface of the object.

$$\therefore \qquad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\nabla' \times \mathbf{M}(\mathbf{r}')}{r} d\tau' + \frac{\mu_0}{4\pi} \oint \frac{\mathbf{M}(\mathbf{r}') \times \hat{\mathbf{n}}}{r} \, da'$$

where $\hat{\mathbf{n}}$ is the normal unit vector of the area element $d\mathbf{a}'$.

$$\therefore \qquad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\nabla' \times \mathbf{M}(\mathbf{r}')}{\mathscr{r}} d\tau' + \frac{\mu_0}{4\pi}\oint \frac{\mathbf{M}(\mathbf{r}') \times \hat{\mathbf{n}}}{\mathscr{r}}\, da'$$

The potential is equivalent to that due to a volume current density $\mathbf{J_b}$ and surface current density $\mathbf{K_b}$, i.e.,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_v \frac{\mathbf{J_b}(\mathbf{r}')}{\mathscr{r}}\, \mathrm{d}\tau' + \frac{\mu_0}{4\pi}\oint_s \frac{\mathbf{K_b}(\mathbf{r}')}{\mathscr{r}} da'$$

where
$$\mathbf{J_b} = \nabla \times \mathbf{M}$$

is the volume bound current density
and
$$\mathbf{K_b} = \mathbf{M} \times \hat{\mathbf{n}}$$

is the surface bound current density.

To prove: $$\int_{v} \nabla' \times \left(\frac{\mathbf{M}(\mathbf{r}')}{\mathscr{r}} \right) d\tau' = -\oint_{s} \frac{1}{\mathscr{r}} \mathbf{M}(\mathbf{r}') \times d\mathbf{a}'$$

Consider a vector field $\mathbf{v}$ and a constant vector $\mathbf{c}$.

By divergence theorem,

$$\int_{v} \nabla' \cdot (\mathbf{v} \times \mathbf{c}) \, \mathrm{d}\tau' = \oint_{s} (\mathbf{v} \times \mathbf{c}) \cdot d\mathbf{a}'$$

$$\therefore \quad \int_{v} [\mathbf{c} \cdot (\nabla' \times \mathbf{v}) - \mathbf{v} \cdot (\nabla' \times \mathbf{c})] \, d\tau' = \oint_{s} \mathbf{c} \cdot (d\mathbf{a}' \times \mathbf{v})$$

where we have used the vector identities:

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

$$\therefore \int_v [\mathbf{c}\cdot(\nabla'\times\mathbf{v}) - \mathbf{v}\cdot(\nabla'\times\mathbf{c})]\,d\tau' = \oint_s \mathbf{c}\cdot(d\mathbf{a}'\times\mathbf{v})$$

Since $\mathbf{c}$ is a constant vector,

$$\nabla'\times\mathbf{c} = 0$$

$$\therefore \mathbf{c}\cdot\int_v (\nabla'\times\mathbf{v})\,d\tau' = \mathbf{c}\cdot\left[-\oint_s \mathbf{v}\times d\mathbf{a}'\right]$$

Because $\mathbf{c}$ is an arbitrary constant vector, therefore,

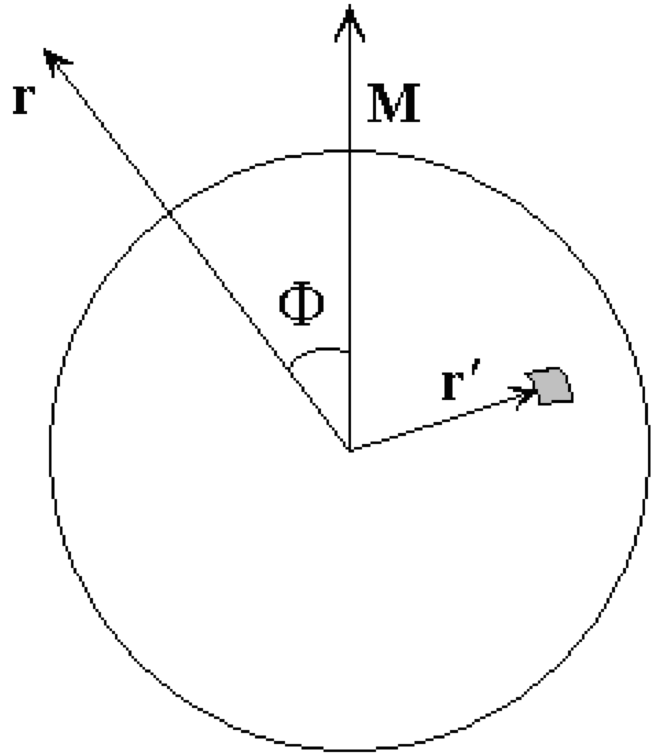
$$\int_v (\nabla'\times\mathbf{v})\,d\tau' = -\oint_s \mathbf{v}\times d\mathbf{a}'$$

The proof is completed by setting $\mathbf{v} = \frac{1}{r}\mathbf{M}(\mathbf{r}')$

Example:

Find the magnetic field of a uniformly magnetized sphere.

Solution:



$$\mathbf{J_b} = \nabla' \times \mathbf{M} = \mathbf{0}$$

$$\mathbf{K_b} = \mathbf{M} \times \hat{\mathbf{n}} = \mathbf{M} \times \hat{\mathbf{r}}'$$

Hence, $\mathbf{A}(\mathbf{r}) = \dfrac{\mu_0}{4\pi} \displaystyle\int \frac{\mathbf{M} \times \hat{\mathbf{r}}'}{\mathscr{r}} da'$

$$= \frac{\mu_0}{4\pi} \mathbf{M} \times \int \frac{\hat{\mathbf{r}}'}{\mathscr{r}} da'$$

For $r > R$,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\mathbf{M}\times\frac{4\pi R^3}{3r^3}\mathbf{r}$$

$$= \frac{\mu_0}{4\pi r^3}\left(\frac{4\pi}{3}R^3\mathbf{M}\right)\times\mathbf{r}$$

$$= \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}$$

It can be shown that (supplementary notes)

$$\oint_S \frac{\hat{\mathbf{r}}'}{|\mathbf{r}-\mathbf{r}'|}da' = \begin{cases} \frac{4\pi}{3}\mathbf{r} & \text{if } r < R \\ \frac{4\pi R^3}{3r^3}\mathbf{r} & \text{if } r > R \end{cases}$$

where $$\mathbf{m} = \mathbf{M}\times\frac{4\pi}{3}R^3$$

is the total dipole moment of the sphere.

So, outside the sphere, the vector potential, and hence the B-field, is exactly the same as if all the dipole moments were at the center, giving rise to an ideal dipole

$$\mathbf{m} = \mathbf{M} \cdot \frac{4\pi}{3} R^3$$

at the origin.

Accordingly, the B-field outside the sphere is

$$\mathbf{B} = \frac{\mu_0}{4\pi} \frac{1}{r^3} \left[3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m} \right]$$

For $r < R$,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\mathbf{M}\times\frac{4\pi}{3}\mathbf{r} = \frac{\mu_0}{3}\mathbf{M}\times\mathbf{r}$$

$$\mathbf{B} = \nabla\times\mathbf{A}$$

$$= \frac{\mu_0}{3}\nabla\times(\mathbf{M}\times\mathbf{r})$$

It can be shown that (supplementary notes)

$$\oint_S \frac{\hat{\mathbf{r}}'}{|\mathbf{r}-\mathbf{r}'|}da' = \begin{cases} \frac{4\pi}{3}\mathbf{r} & \text{if } r < R \\ \frac{4\pi R^3}{3r^3}\mathbf{r} & \text{if } r > R \end{cases}$$

But $\nabla\times(\mathbf{M}\times\mathbf{r}) = (\mathbf{r}\cdot\nabla)\mathbf{M} - (\mathbf{M}\cdot\nabla)\mathbf{r} + \mathbf{M}(\nabla\cdot\mathbf{r}) - \mathbf{r}(\nabla\cdot\mathbf{M})$

Since $\mathbf{M}$ is a constant vector, $\nabla\times(\mathbf{M}\times\mathbf{r}) = \mathbf{M}(\nabla\cdot\mathbf{r}) - (\mathbf{M}\cdot\nabla)\mathbf{r}$

$$= 3\mathbf{M} - (\mathbf{M}\cdot\nabla)\mathbf{r}$$

And $(\mathbf{M}\cdot\nabla)\mathbf{r} = \left(M_x\frac{\partial}{\partial x} + M_y\frac{\partial}{\partial y} + M_z\frac{\partial}{\partial z}\right)(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) = \mathbf{M}$

$$\therefore \nabla\times(\mathbf{M}\times\mathbf{r}) = 2\mathbf{M}$$

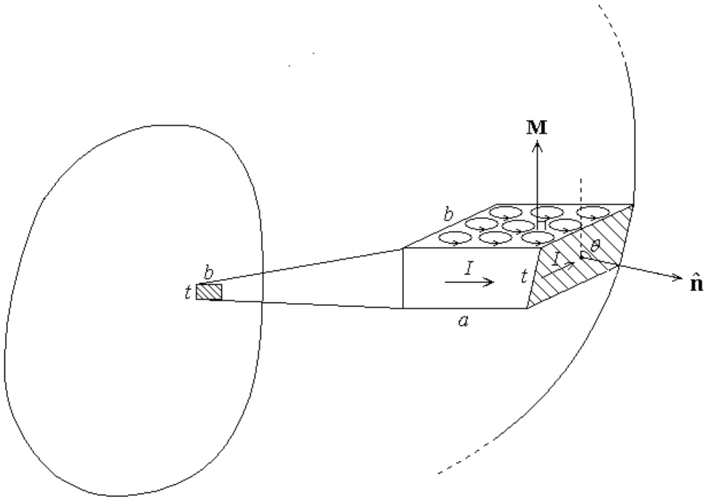
$$\therefore\ \mathbf{B} = \frac{2}{3}\mu_0\mathbf{M}$$

Magnetic Field in Matters

Physical Interpretation
of Bound Currents

Consider a small area element on the surface of a magnetized object with length b and width t.

Consider a very small volume element as shown above, with length a, width b and thickness $t\sin\theta$.



The region is so small that $\mathbf{M}$ can be taken as constant.

There are tiny current loops inside the volume due to magnetization.

Inside the volume, the current at a point due to adjacent loops cancel each other.

The net result is a current flowing on the surface.
The magnetic dipole moment is therefore

$$m = Iab$$

where I is the surface current.

But the net dipole moment must be the same as that of the total contribution of all the dipoles inside the volume.

$$\therefore Iab = Mabt \sin\theta$$

$$I = Mt \sin\theta$$

By the definition of surface current density,

$$K = \frac{I}{t} = M \sin\theta \qquad (\because I = Mt\sin\theta)$$

$$\therefore \mathbf{K} = \mathbf{M} \times \hat{\mathbf{n}}$$

Physical Interpretation of volume bound current:

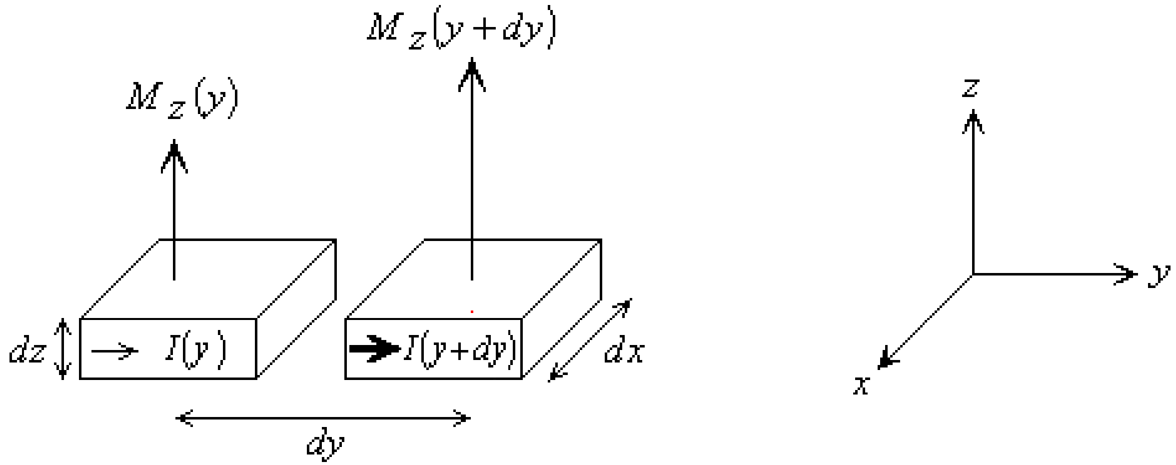
When $\mathbf{M}$ is not uniform, the current carried by adjacent loops cannot cancel each other.

For example, consider the *x*-component of the currents.

Since M_x is responsible for currents flowing on the plane parallel to the *y*-*z* plane, we need not consider M_x

Since M_x is responsible for currents flowing on the plane parallel to the *y*-*z* plane, we need not consider M_x

M_z:



the current on the surface of the two adjacent volume elements are of different magnitudes and cannot cancel each other

$$I(y)dxdy = M_z(y)dxdydz$$

$$\therefore I(y) = M_z(y)dz$$

If M_z is non-uniform in the *y*-direction, i.e. $\dfrac{\partial M_z}{\partial y} \neq 0$

Similarly, $$I(y+dy) = M_z(y+dy)dz$$

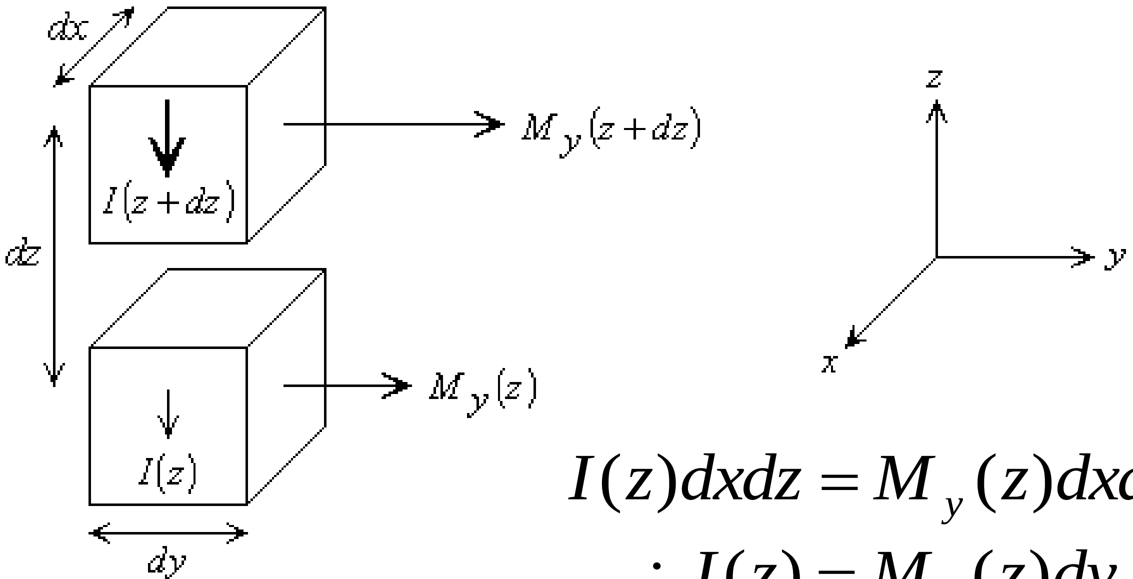
The net current flowing in the *x*-direction is

$$I = I(y+dy) - I(y) = [M_z(y+dy) - M_z(y)]dz$$

$\therefore$ By definition of volume current density,

$$(\mathbf{J_b})_x = \frac{I}{dydz} = \frac{M_z(y+dy) - M_z(y)}{dy} = \frac{\partial M_z}{\partial y}$$

The variation of M_y in the *z*-direction can also give rise to a current in the *x*-direction.



$$I(z)dxdz = M_y(z)dxdydz$$

$$\therefore I(z) = M_y(z)dy$$

Similarly, $$I(z+dz) = M_y(z+dz)dy$$

The net current flowing in the *x*-direction is

$$I = -I(z+dz) + I(z) = -[M_y(z+dz) - M_y(z)]dy$$

By definition of volume current density,

$$\therefore \quad (\mathbf{J_b})_x = \frac{I}{dydz} = -\frac{M_y(z+dz) - M_y(z)}{dz} = -\frac{\partial M_y}{\partial z}$$

Taking into account the two contributions,

$$(\mathbf{J_b})_x = \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}$$

Similarly, one can prove that

$$(\mathbf{J_b})_y = \frac{\partial M_x}{\partial z} - \frac{\partial M_z}{\partial x}$$

$$(\mathbf{J_b})_z = \frac{\partial M_x}{\partial y} - \frac{\partial M_y}{\partial z}$$

$$\boxed{\therefore \mathbf{J_b} = \nabla \times \mathbf{M}}$$

Magnetic Field in Matters

The Auxiliary Field H

Inside matters,

$$\mathbf{J} = \mathbf{J_f} + \mathbf{J_b}$$

$\mathbf{J}$	:	*Total current density*
$\mathbf{J_f}$	:	*Free current density*
$\mathbf{J_b}$	:	*Bound current density (due to magnetization)*

By Ampere's law,

$$\begin{aligned}\nabla \times \mathbf{B} &= \mu_0 \mathbf{J} \\ &= \mu_0 \mathbf{J_f} + \mu_0 \mathbf{J_b} \\ &= \mu_0 \mathbf{J_f} + \mu_0 \nabla \times \mathbf{M}\end{aligned}$$

$$\therefore \nabla \times \left(\frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \right) = \mathbf{J_f}$$

Define the H-field,

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$$

$$\therefore \nabla \times \mathbf{H} = \mathbf{J_f} \quad \text{(Ampere's law of } \mathbf{H}\text{)}$$

In integral form,

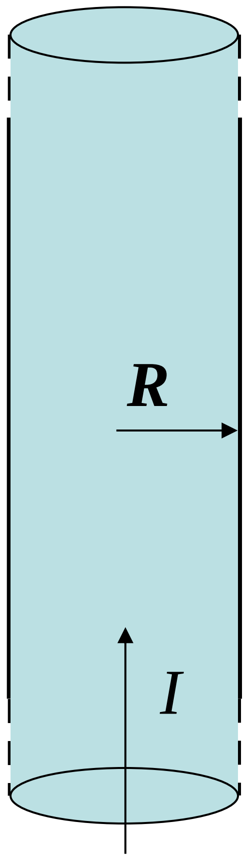
$$\oint \mathbf{H} \cdot d\mathbf{l} = \mathbf{I}_{f\text{enc}}$$

where $\mathbf{I}_{f\text{enc}}$ is the free current enclosed.

Example:

A long copper rod of radius R carries a uniformly distributed (free) current I .

Find $\mathbf{H}$ inside and outside the rod.

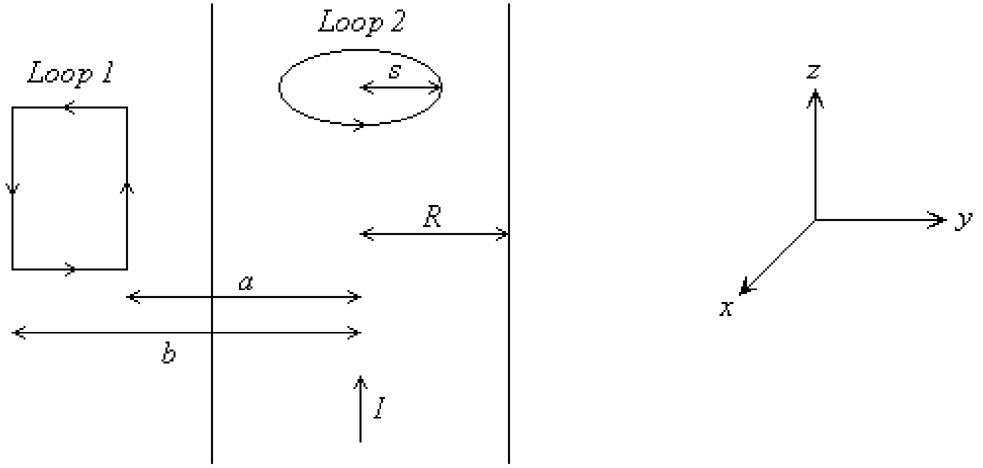


Answer:

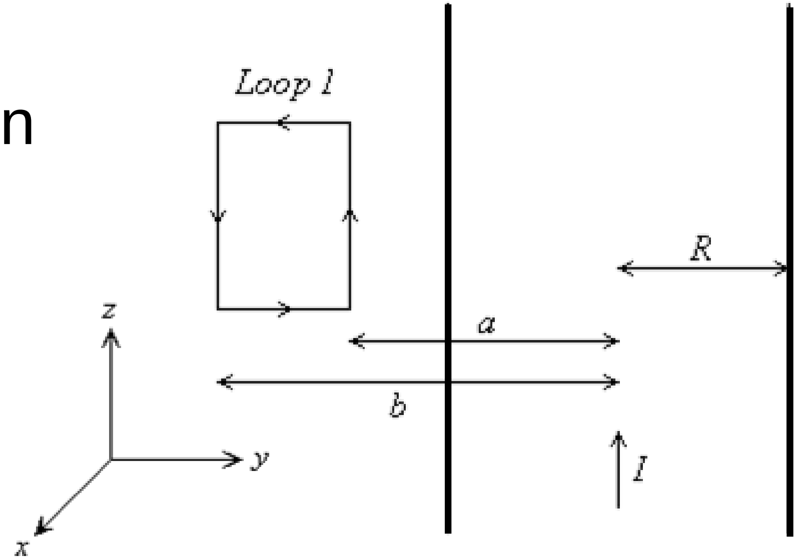
The argument is similar to that of a solenoid.

First, the **H** field has no radial component.

And the field depends only on *s*, but not on z and ϕ .



In addition, if one considers the amperian loop 1, it can be shown that $\mathbf{H}_z$ is constant.



This is true no matter where the loop is located, because unlike the case of a solenoid, the current is now flowing in the *z*-direction and loop 1 always encloses no current.

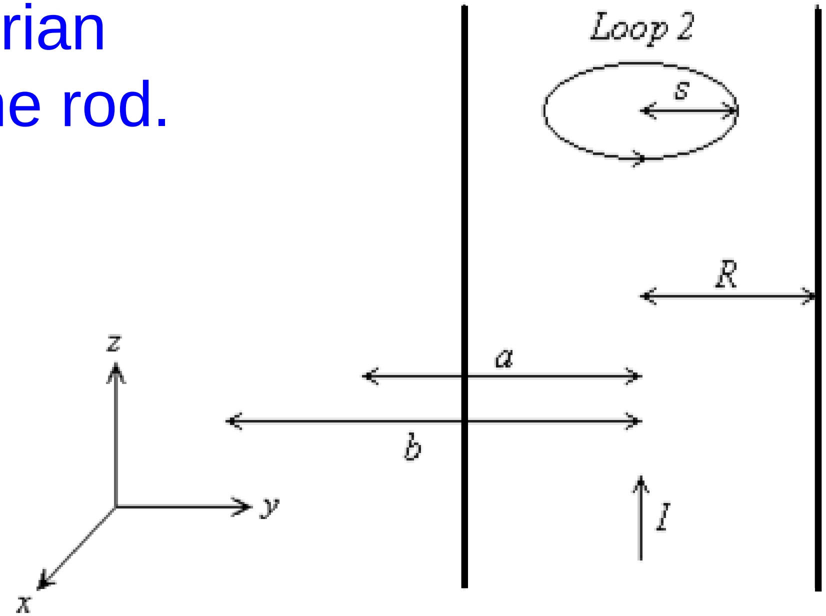
$$\therefore H_z(a) = H_z(b) \qquad \forall a, b > 0$$

However, we know that $\mathbf{H} = \mathbf{0}$ at infinity, $\therefore H_z = 0$

Therefore, the $\mathbf{H}$ field only has the $\hat{\boldsymbol{\varphi}}$ -component.

Now consider the circular amperian loop 2 coaxial with the axis of the rod.
By Ampere's law,
For $s > R$,

$$H_\phi(s) \cdot 2\pi s = I$$
$$\therefore \mathbf{H} = \frac{I}{2\pi s}\hat{\boldsymbol{\varphi}}$$



For $s < R$,

$$H_\phi(s) \cdot 2\pi s = I \cdot \frac{\pi s^2}{\pi R^2} = I\,\frac{s^2}{R^2}$$
$$\therefore \mathbf{H} = \frac{Is}{2\pi R^2}\hat{\boldsymbol{\varphi}}$$

Since **M** is unknown, the **B**-field inside the rod cannot be determined. However, outside the rod,

$$\mathbf{M} = \mathbf{0}$$

$$\therefore \mathbf{B} = \mu_0 \mathbf{H}$$

$$= \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\varphi}} \quad \text{,for} \quad s \geq R$$

Note, that although $\nabla \times \mathbf{H} = \mathbf{J_f}$ looks similarly to $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$

However, $\nabla \cdot \mathbf{B} = 0$ while,

$$\nabla \cdot \mathbf{H} = \nabla \cdot \left(\frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \right)$$

$$= -\nabla \cdot \mathbf{M}$$,which may not vanish.

Both $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ and $\nabla \cdot \mathbf{B} = 0$

have been employed to determine the B-field.

It is not valid to perform the simple substitution,

$$\mathbf{H} \leftrightarrow \frac{1}{\mu_0} \mathbf{B}$$

$$\mathbf{J_f} \leftrightarrow \mathbf{J}$$

Magnetic Field in Matters

Linear Media

For some materials, the magnetization is directly proportional to the applied field.

$$\mathbf{M} \propto \mathbf{B}$$

Let $\mathbf{M} = k\mathbf{B}$,

$$\therefore \mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$$

$$= \left(\frac{1}{\mu_0 k} - 1\right)\mathbf{M}$$

$$\therefore \mathbf{M} = \frac{\mu_0 k}{1 - \mu_0 k}\mathbf{H}$$

i.e., $\mathbf{M}$ is also proportional to $\mathbf{H}$.

By convention, the magnetic susceptibility χ_m is defined by,

$$\mathbf{M} = \chi_m \mathbf{H}$$

$$\therefore \mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}) = \mu_0 (1 + \chi_m)\mathbf{H}$$

$$= \mu \mathbf{H}$$

where $\mu = \mu_0 (1 + \chi_m)$ is called the *permeability* of the material

Recall that μ_0 is called the permeability of free space

Besides, $\dfrac{\mu}{\mu_0} = 1 + \chi_m$ is called the relative permeability

Note: Recall that we cannot simply substitute

$$\mathbf{H} \leftrightarrow \frac{1}{\mu_0}\mathbf{B}$$

$$\mathbf{J_f} \leftrightarrow \mathbf{J}$$

Since $\nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M}$ may be non-zero,

In linear media,

$$\mathbf{H} = \frac{1}{\mu}\mathbf{B}$$

$$\therefore \nabla \cdot \mathbf{H} = \frac{1}{\mu}\nabla \cdot \mathbf{B} = 0$$

It seems that the substitution becomes possible.
In fact, this is true if the entire space is filled by a single medium.
In other words,

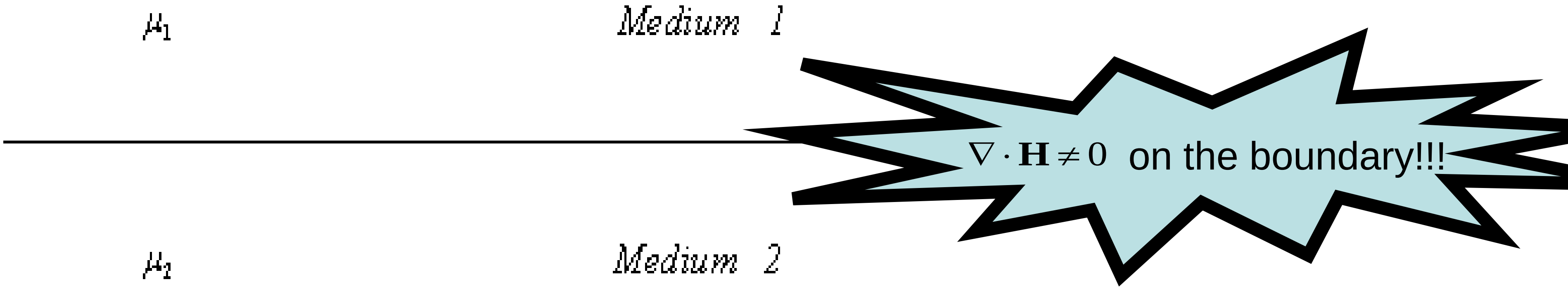
μ is constant in the entire space.

However, if μ is position dependent, then

$$\nabla \cdot \mathbf{H} = \nabla \cdot \left(\frac{1}{\mu} \mathbf{B} \right) \neq \frac{1}{\mu} \nabla \cdot \mathbf{B}$$

may be non-zero.

For example:



In this case, we have to match the boundary conditions.

Magnetic Field in Matters

Boundary Conditions

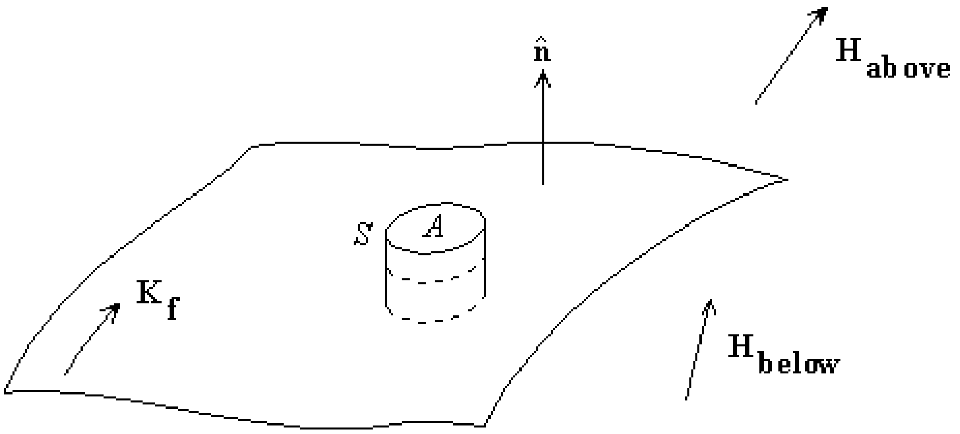
Consider a surface with free current density $\mathbf{K}_f$

Consider a pillbox Gaussian surface as shown.

Since $\nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M}$

or in integral form,

$$\oint_S \mathbf{H} \cdot d\mathbf{a} = -\oint_S \mathbf{M} \cdot d\mathbf{a}$$



we have,

$$H^{\perp}_{\text{above}} - H^{\perp}_{\text{below}} = -(M^{\perp}_{\text{above}} - M^{\perp}_{\text{below}})$$

For the parallel components, since

$\nabla \times \mathbf{H} = \mathbf{J_f}$ is similar to $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$

Recall that we have derived from the Ampere's law that,

$$\mathbf{B}^{//}_{\text{above}} - \mathbf{B}^{//}_{\text{below}} = \mu_0 \mathbf{K} \times \hat{\mathbf{n}}$$

Therefore, similarly

$$\mathbf{H}^{//}_{\text{above}} - \mathbf{H}^{//}_{\text{below}} = \mathbf{K}_f \times \hat{\mathbf{n}}$$

In particular, if $\mathbf{K}_f = 0$

then

$$\mathbf{H}^{//}_{\text{above}} = \mathbf{H}^{//}_{\text{below}}$$

In linear media,

$$\mathbf{H} = \frac{1}{\mu}\mathbf{B}$$

At the interface of two linear media,

$$\begin{cases} B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}} \\ \dfrac{1}{\mu_{\text{above}}}\mathbf{B}^{//}_{\text{above}} = \dfrac{1}{\mu_{\text{below}}}\mathbf{B}^{//}_{\text{below}} \end{cases}$$