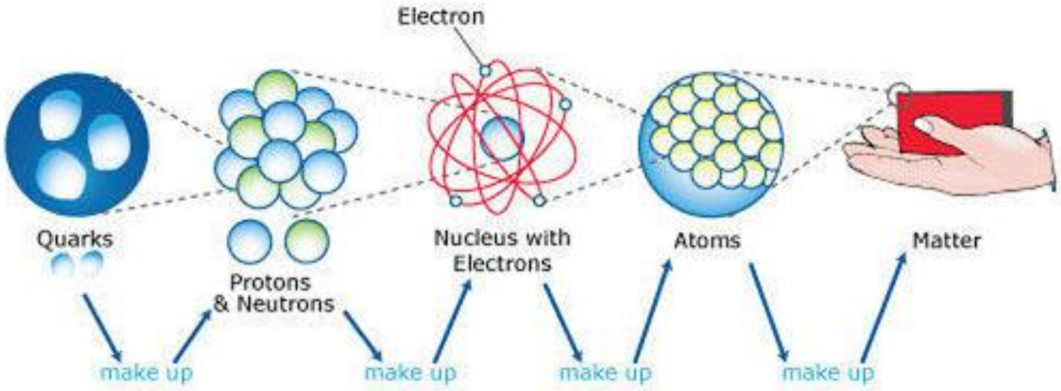
Electric Fields inside Matter

- Inside a medium, unlike in vacuum, there are a lot of charged particles, e.g., electrons, protons, etc.



- In principle, matter is just a form of source distribution in vacuum, and we can apply what we learned to obtain the fields, provided that the ρ due to matter inside the medium is known

- Note, that unlike free source distributions, the sources inside matter are due to objects with atomic dimensions, which are very small compared to macroscopic length scales
- Hence one can use the multipole expansion and keep only the leading order term

- For the electric field, because the atoms and molecules carry no net charge (monopole moment), hence the leading order term is also the **dipole term**
- This makes it possible to consider matter as consisting of a large number of pure electric dipoles

Electric Fields inside Matter

~~Anti-matter~~ ?

~~Dark matter~~ ??

Matter

Conductors

- Unlimited supply of free charges

Dielectrics

- Insulators. Polarized when external E field is applied

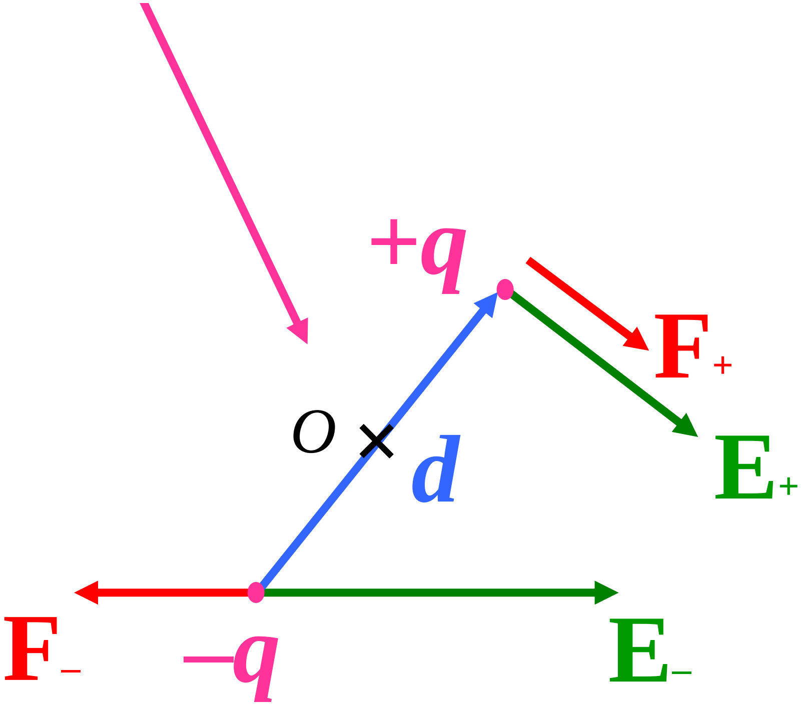
Electric Fields in Matter

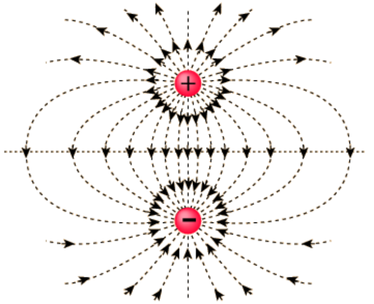
Effect of External Field on Matter

Force and torque on a dipole in external E field:

External force:

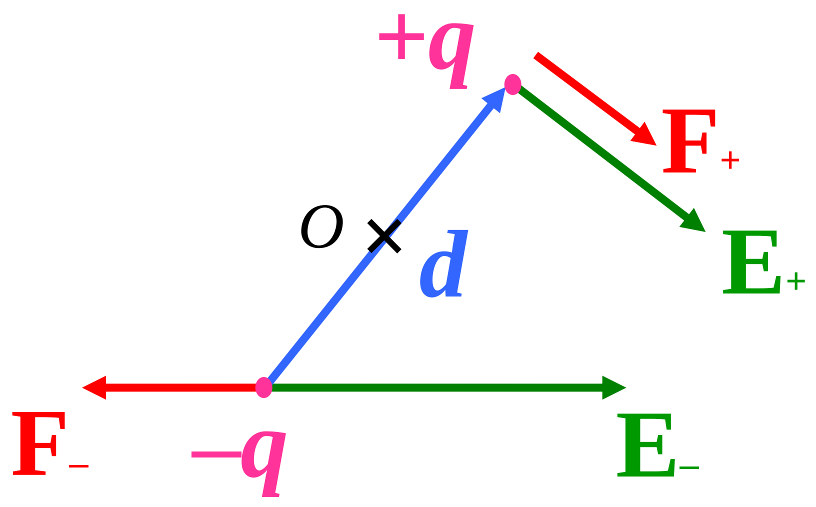
$$\mathbf{F} = \mathbf{F}_+ + \mathbf{F}_-$$
$$= q\left(\mathbf{E}_+ - \mathbf{E}_-\right)$$





Force and torque on a dipole in external E field:

External force:



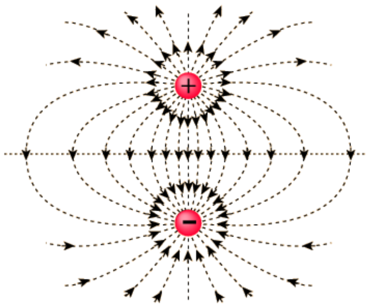
$$\left(\mathbf{E}_+ - \mathbf{E}_-\right)_x = E_x\left(\mathbf{r}_+\right) - E_x\left(\mathbf{r}_-\right)$$

$$= \left[E_x(O) + \nabla E_x(O)\cdot\frac{\mathbf{d}}{2}\right] - \left[E_x(O) + \nabla E_x(O)\cdot\left(-\frac{\mathbf{d}}{2}\right)\right]$$

$$= \nabla E_x(O)\cdot\mathbf{d}$$

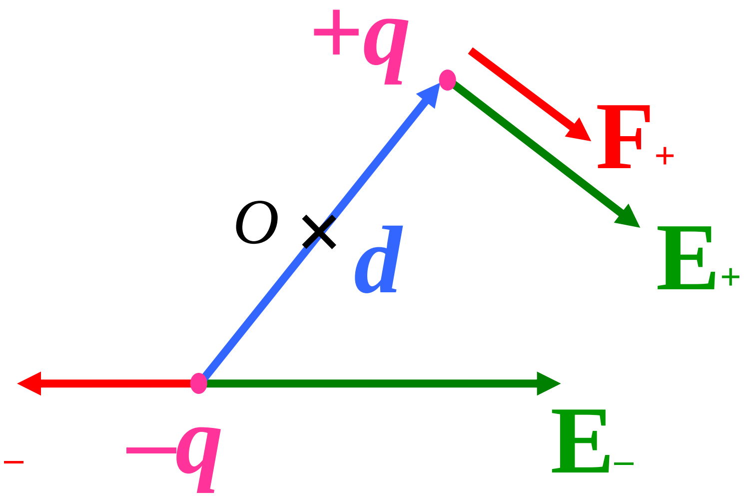
$$= d_x\frac{\partial E_x}{\partial x} + d_y\frac{\partial E_x}{\partial y} + d_z\frac{\partial E_x}{\partial z}$$

$$= \left(\mathbf{d}\cdot\nabla\right)E_x$$



Force and torque on a dipole in external E field:

External force:

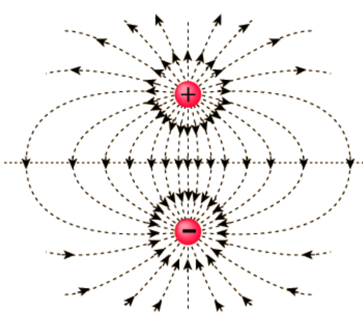


$$\left(\mathbf{E}_+ - \mathbf{E}_-\right)_x = \left(\mathbf{d}\cdot\nabla\right)E_x$$

$$\left(\mathbf{E}_+ - \mathbf{E}_-\right)_y = \left(\mathbf{d}\cdot\nabla\right)E_y$$

$$\left(\mathbf{E}_+ - \mathbf{E}_-\right)_z = \left(\mathbf{d}\cdot\nabla\right)E_z$$

$$\begin{aligned}\mathbf{E}_+ - \mathbf{E}_- &= \hat{\mathbf{x}}\left(\mathbf{d}\cdot\nabla\right)E_x + \hat{\mathbf{y}}\left(\mathbf{d}\cdot\nabla\right)E_y + \hat{\mathbf{z}}\left(\mathbf{d}\cdot\nabla\right)E_z \\ &= \left(\mathbf{d}\cdot\nabla\right)\left(E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} + E_z\hat{\mathbf{z}}\right) \\ &= \left(\mathbf{d}\cdot\nabla\right)\mathbf{E}\end{aligned}$$



Force and torque on a dipole in external E field:

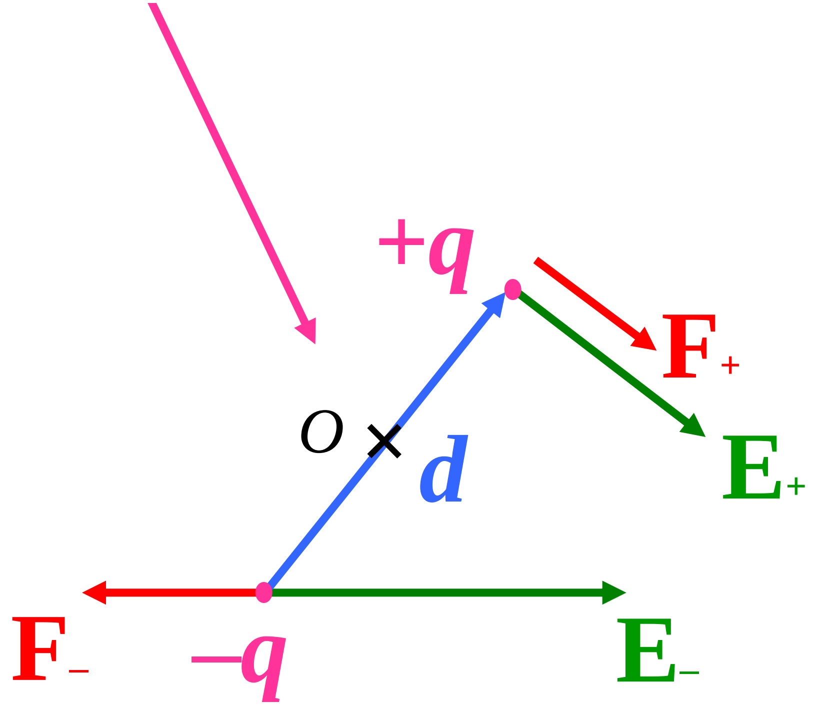
External force:

$$\mathbf{F} = \mathbf{F}_+ + \mathbf{F}_-$$

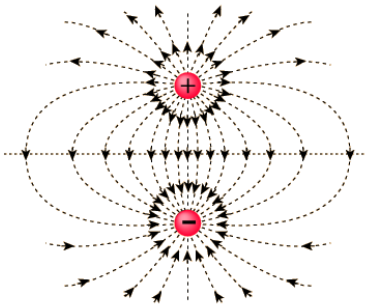
$$= q\left(\mathbf{E}_+ - \mathbf{E}_-\right)$$

$$= q\left(\mathbf{d} \cdot \nabla\right)\mathbf{E}$$

$$\boxed{\mathbf{F} = \left(\mathbf{p} \cdot \nabla\right)\mathbf{E}}$$



In particular, in a ***uniform E field***, $\mathbf{F} = \mathbf{0}$.



Force and torque on a dipole in external E field:

Torque:

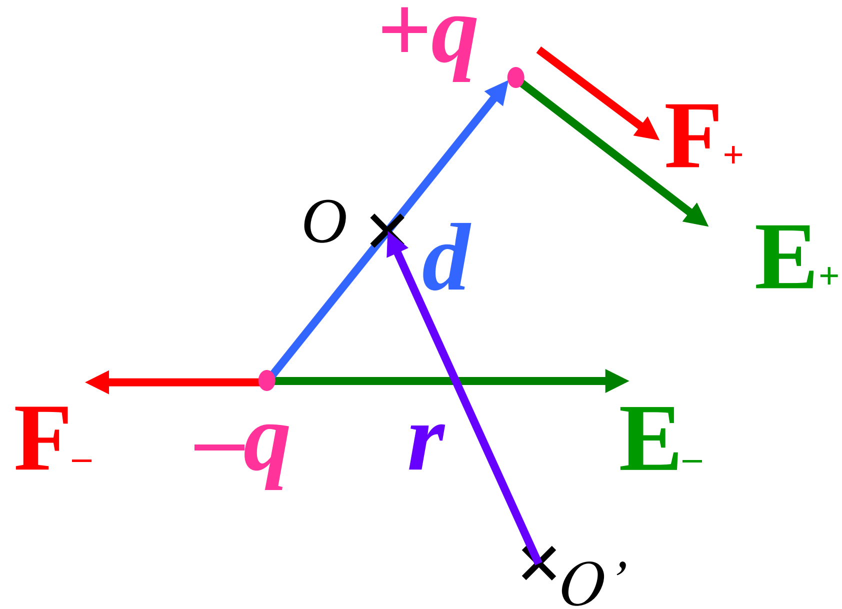
Let the position vector of the center of the dipole, O, be $\mathbf{r}$.
The torque about the origin is

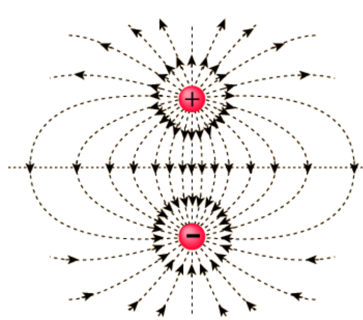
$$\mathbf{N} = \left(\mathbf{r} + \frac{\mathbf{d}}{2}\right) \times \mathbf{F}_+ + \left(\mathbf{r} - \frac{\mathbf{d}}{2}\right) \times \mathbf{F}_-$$

$$= \mathbf{r} \times (\mathbf{F}_+ + \mathbf{F}_-) + \frac{\mathbf{d}}{2} \times (\mathbf{F}_+ - \mathbf{F}_-)$$

$$= \mathbf{r} \times (\mathbf{F}_+ + \mathbf{F}_-) + q\mathbf{d} \times \frac{\mathbf{E}_+ + \mathbf{E}_-}{2}$$

$$= \mathbf{r} \times \mathbf{F} + \mathbf{p} \times \frac{\mathbf{E}_+ + \mathbf{E}_-}{2}$$





For an ideal dipole, $d \to 0$ and $\mathbf{E}_+, \mathbf{E}_- \to \mathbf{E}$ which is the E field at O

Therefore, $$\boxed{\mathbf{N} = \mathbf{r} \times \mathbf{F} + \mathbf{p} \times \mathbf{E}}$$

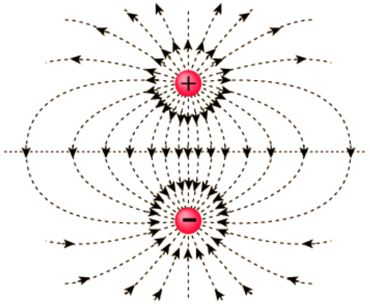
Note that there is torque even when the field is uniform and $\mathbf{F} = \mathbf{0}$:

$$\mathbf{N} = \mathbf{p} \times \mathbf{E}$$

In addition, the torque about the center is always

$$\boxed{\mathbf{N} = \mathbf{p} \times \mathbf{E}}$$

even in the case of non-uniform fields.

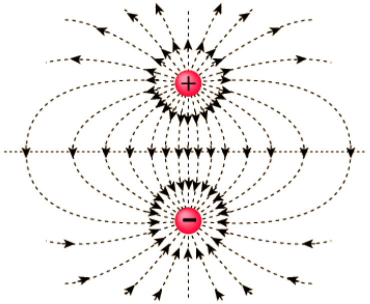


Torque about center:

$$\boxed{\mathbf{N} = \mathbf{p} \times \mathbf{E}}$$

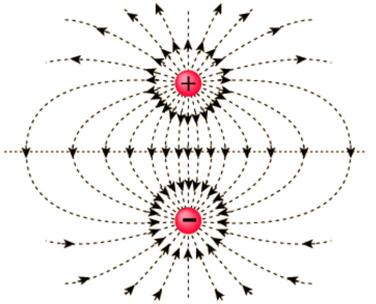
- The torque is zero when $\mathbf{p}$ is pointing at the same direction as $\mathbf{E}$, or in the opposite direction.
- When $\mathbf{p}$ is parallel to $\mathbf{E}$, the dipole is in stable equilibrium.
- When $\mathbf{p}$ is anti-parallel to $\mathbf{E}$, it is unstable.

Therefore, the torque tends to align the dipole with the electric field.



The energy of an ideal dipole $\mathbf{p}$ due to the torque exerted by the electric field is:

$$\boxed{U = -\mathbf{p} \cdot \mathbf{E}}$$



Macroscopic dipole moments created by external E field:

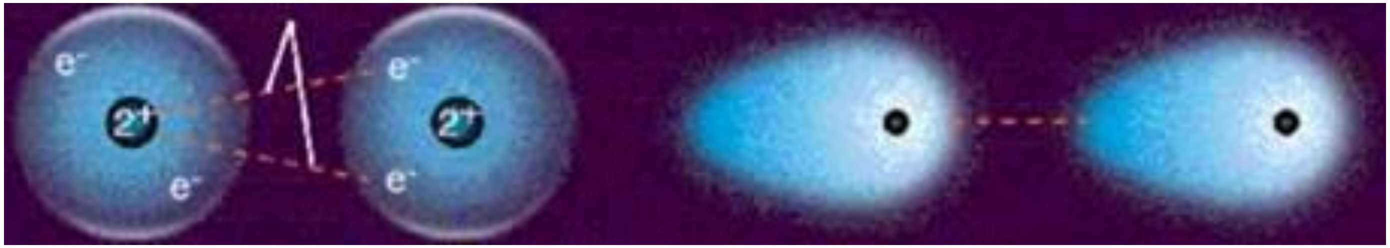
When an ***external E field*** is applied on a dielectrics, which originally shows no macroscopic dipole moment, a ***net dipole moment*** can be observed macroscopically.

There are mainly 2 different mechanisms giving rise to the polarization observed:

1. **Induced dipoles**
2. **Alignment of polar molecules**

1. Induced dipoles

- Atoms or molecules with no dipole moment originally.
- Applied $\mathbf{E}$ field push the nuclei and the electron clouds in opposite directions.
- Creates net dipole moments.



The dipole moment induced is:

- along the direction of the applied field
- with magnitude proportional to that of the field when the field is weak

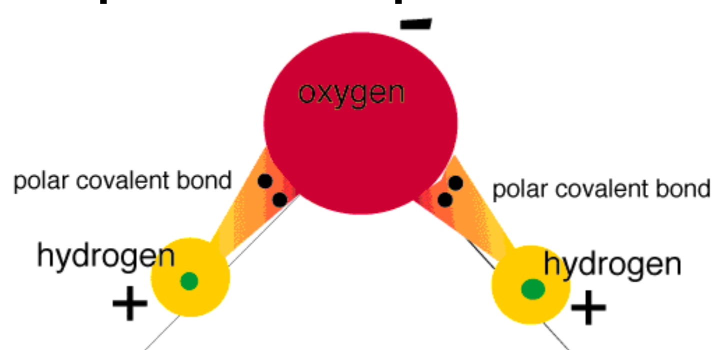
In other words,

$$\mathbf{p} = \alpha \mathbf{E}$$

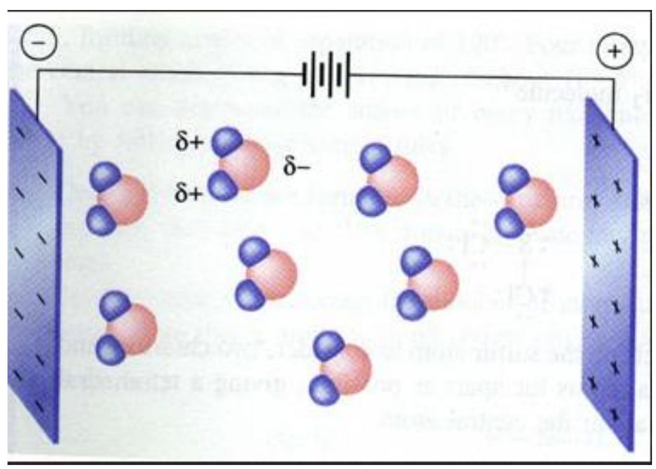
atomic polarizability

2. Alignment of polar molecules

- Some molecules possess permanent dipole moments.



- The applied field gives rise to a torque



- The torque tends to align the dipoles with $\boldsymbol{E}$ until the torque = $\mathbf{0}$ and the energy $\mathbf{U} = -\mathbf{p} \cdot \mathbf{E}$ is minimum.

To a first order approximation, we also assume that the dipole moment is

- with magnitude proportional to that of the field when the field is weak

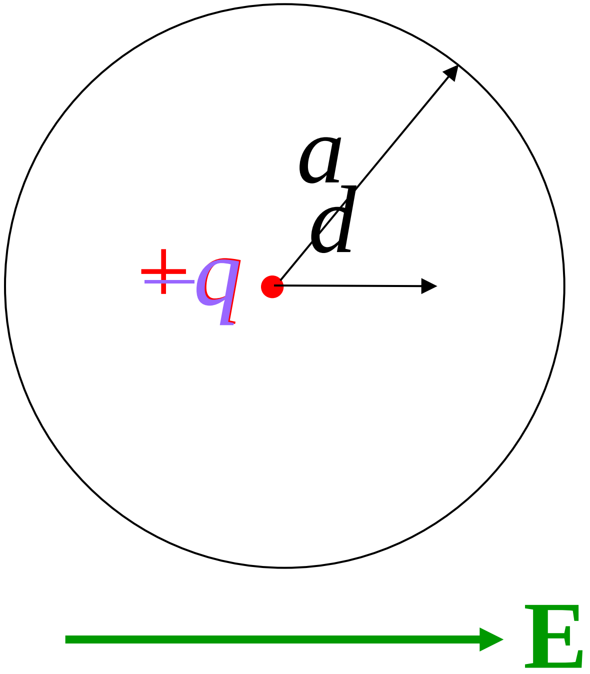
Polarization

- The polarization, $\mathbf{P}$, is defined by the dipole moment per unit volume:

$$\boxed{\mathbf{P} = \text{ dipole moment per unit volume}}$$

Example:

A primitive model for an atom consists of a point nucleus (+q) surrounded by a uniformly charged spherical cloud (-q) of radius a. Calculate the atomic polarizability of such an atom.



In equilibrium, the force on the nucleus exerted by the external field and the uniformly charged sphere must cancel.

Recall: *the magnitude of the electric field at a distance d (d<<R) from the center of a uniformly charged sphere, with total charge q,*

is $\dfrac{1}{4\pi\varepsilon_0}\dfrac{qd}{a^3}$ *. So,*

$$E = \frac{1}{4\pi\varepsilon_0}\frac{qd}{a^3}$$

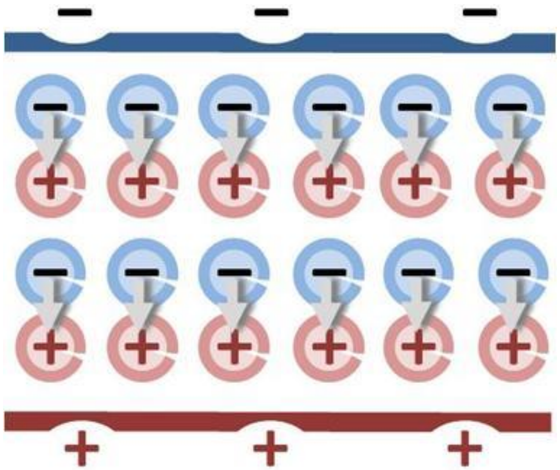
$$p = qd = 4\pi\varepsilon_0 a^3 E$$

$$\alpha = 4\pi\varepsilon_0 a^3 = 3\varepsilon_0 \frac{4\pi a^3}{3} = 3\varepsilon_0 v$$

where v is the volume of the atom.

Electric Fields in Matter

The Field of a Polarized Object



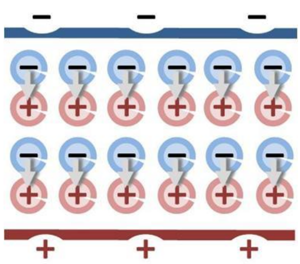
Consider an object with polarization $\mathbf{P}$. Recall that the potential of a pure dipole $\mathbf{p}$ is

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{p}}{\mathscr{r}^2}$$

The potential produced by this object is therefore

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}}\frac{\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{P}(\mathbf{r}')}{\mathscr{r}^2}d\tau'$$

where $\mathcal{V}$ is the volume of the object.



Consider the gradient with respect to $\mathbf{r}'$

$$\nabla'\left(\frac{1}{\mathscr{r}}\right)=\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)=-\nabla_{\mathbf{r}}\left(\frac{1}{\mathscr{r}}\right)=\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

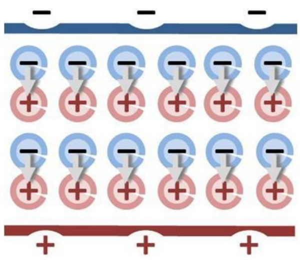
Recall : $\nabla\left(\frac{1}{\mathscr{r}}\right)=\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)=-\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$

Therefore,

$$V(\mathbf{r})=\frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}}\frac{\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{P}(\mathbf{r}')}{\mathscr{r}^2}d\tau'$$

$$=\frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}}\nabla'\left(\frac{1}{\mathscr{r}}\right)\cdot\mathbf{P}(\mathbf{r}')d\tau'$$

$$=\frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}}\left(\nabla'\cdot\left(\frac{\mathbf{P}(\mathbf{r}')}{\mathscr{r}}\right)-\frac{\nabla'\cdot\mathbf{P}(\mathbf{r}')}{\mathscr{r}}\right)d\tau'$$



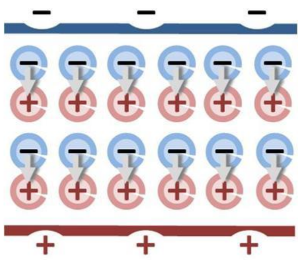
$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}}\left(\nabla'\cdot\left(\frac{\mathbf{P}(\mathbf{r}')}{\imath}\right) - \frac{\nabla'\cdot\mathbf{P}(\mathbf{r}')}{\imath}\right)d\tau'$$

By divergence theorem, the first integral can be turned into a surface integral over the surface of the object, *S*:

$$\frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}}\nabla'\cdot\left(\frac{\mathbf{P}(\mathbf{r}')}{\imath}\right)d\tau' = \frac{1}{4\pi\varepsilon_0}\int_{S}\frac{\mathbf{P}(\mathbf{r}')\cdot\hat{\mathbf{n}}}{\imath}da'$$

So,

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{S}\frac{\mathbf{P}(\mathbf{r}')\cdot\hat{\mathbf{n}}}{\imath}da' + \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}}\frac{-\nabla'\cdot\mathbf{P}(\mathbf{r}')}{\imath}d\tau'$$



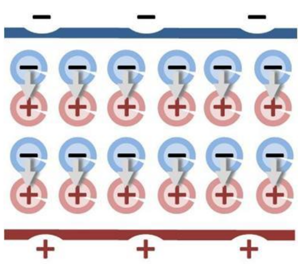
$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\mathbf{P}(\mathbf{r}')\cdot\hat{\mathbf{n}}}{\mathscr{r}}da' + \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}} \frac{-\nabla'\cdot\mathbf{P}(\mathbf{r}')}{\mathscr{r}}d\tau'$$

The field is the same as that of surface **bound charges** σ_b on *S* and volume **bound charges** ρ_b in $\boldsymbol{V}$, with

$$\boxed{\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}} \quad \text{and} \quad \boxed{\rho_b = -\nabla\cdot\mathbf{P}}$$

i.e.

$$V(\mathbf{r}) = \underbrace{\frac{1}{4\pi\varepsilon_0}\int_S \frac{\sigma_b}{\mathscr{r}}ds}_{\frac{1}{4\pi\varepsilon_0}\int_S \frac{\mathbf{P}(\mathbf{r}')\cdot\hat{\mathbf{n}}}{r}ds} + \underbrace{\frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}} \frac{\rho_b}{\mathscr{r}}d\tau'}_{\frac{1}{4\pi\varepsilon_0}\int_V \frac{-\nabla'\cdot\mathbf{P}(\mathbf{r}')}{r}d\tau'}$$

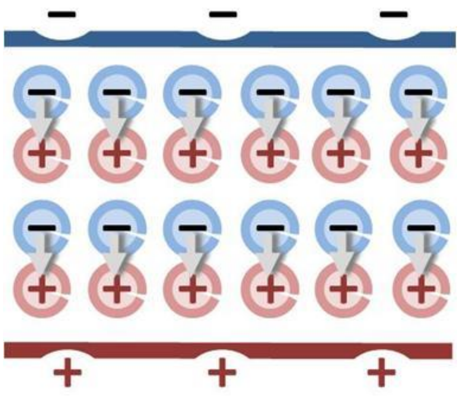


What are bound charges???

Physical interpretation of bound charges:

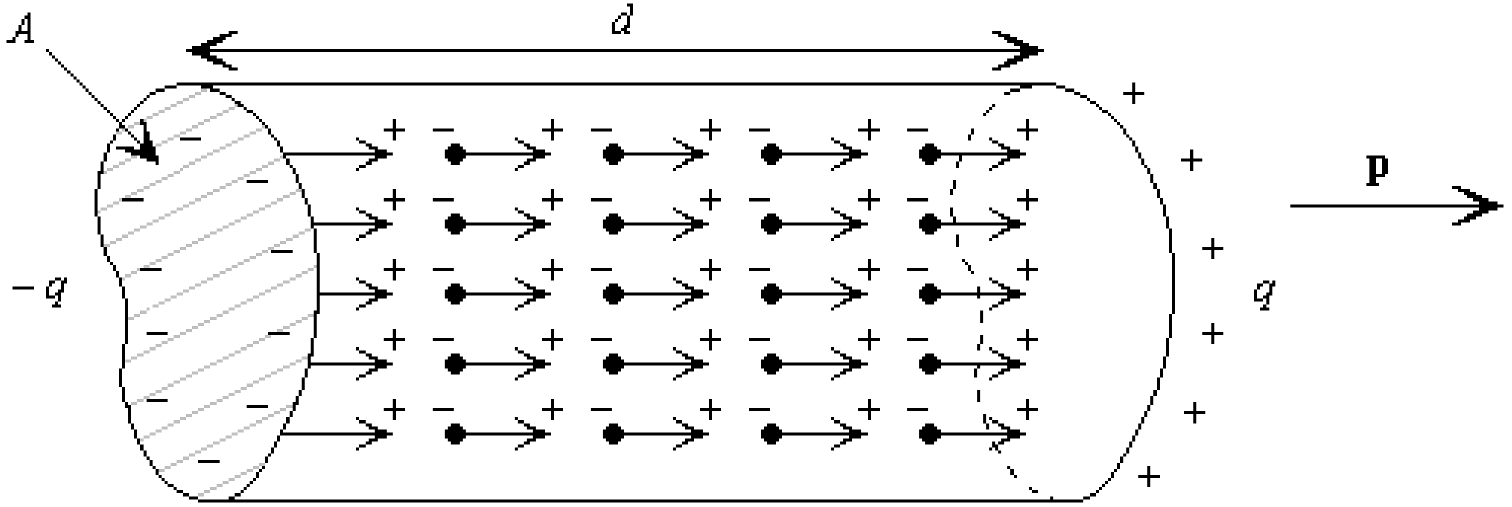
We derived the expressions for the bound charges mathematically. They are not just mathematical tools.

They are ***physically real***.



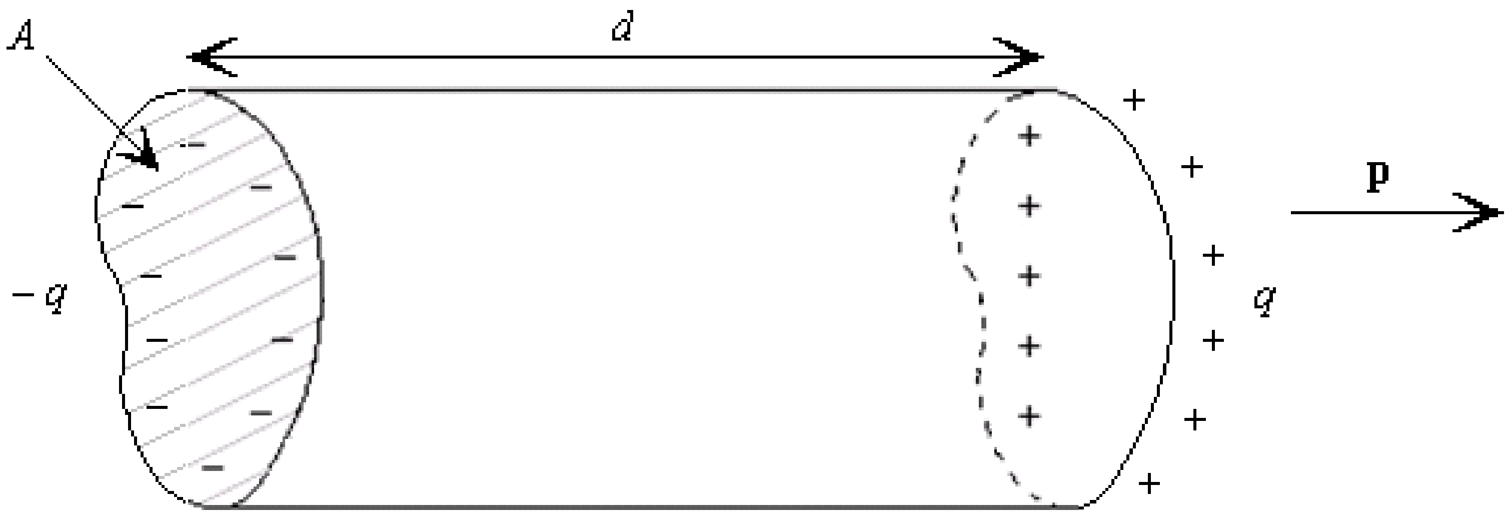
Surface bound charges:

Consider a very small volume element as shown below, which is small enough such that **P** is uniform



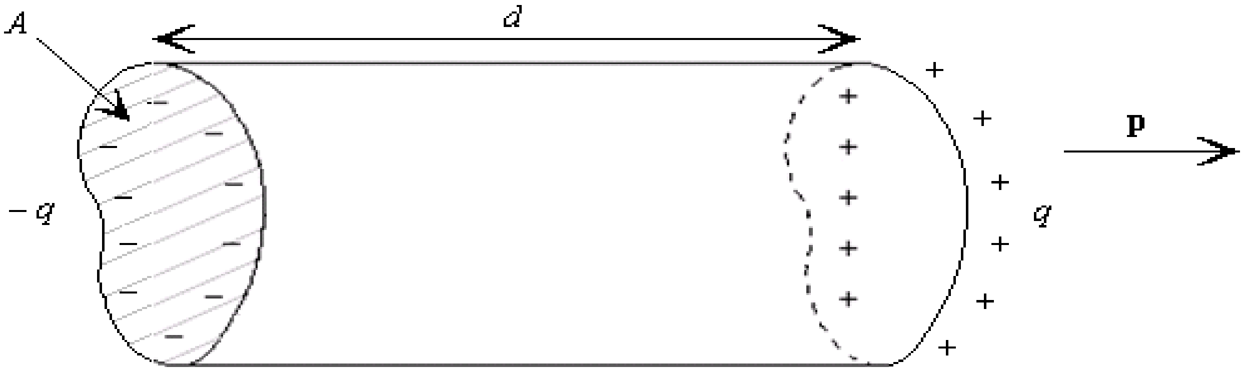
Surface bound charges:

The positive charges at the head of the dipoles cancel with the negative charges at the tail inside the volume, except on the surfaces, where there are no other positive or negative charges to cancel them.



Surface bound charges:

Let the charge be $+q$ and $-q$



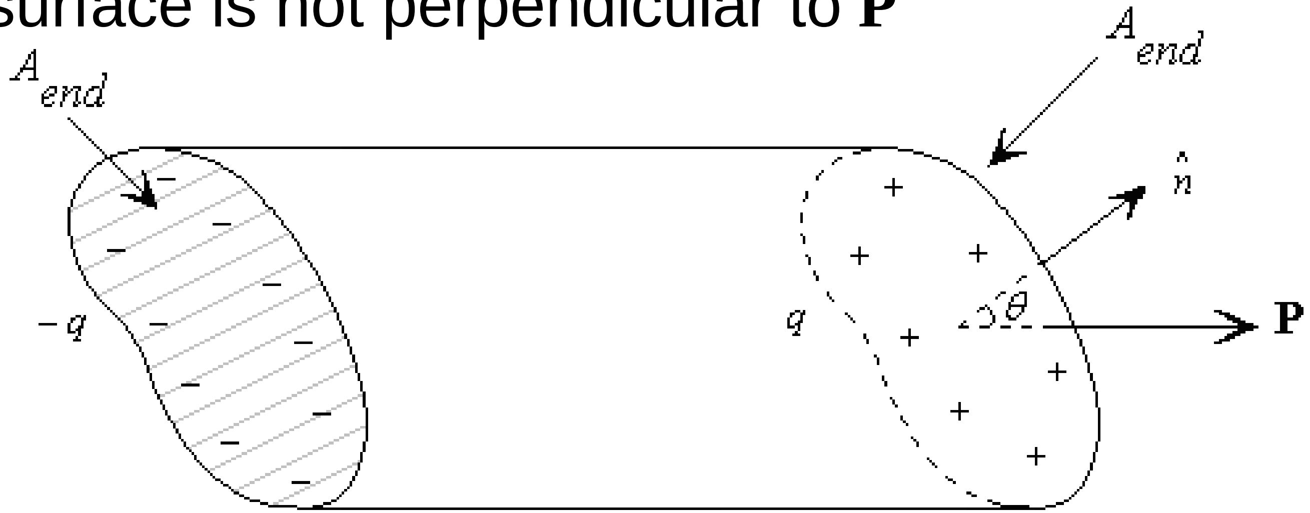
Then, $$p = P \cdot Ad = qd$$

$$q = P \cdot A$$

Therefore, $$\sigma_b = \frac{q}{A} = P$$

Surface bound charges:

If the surface is not perpendicular to **P**

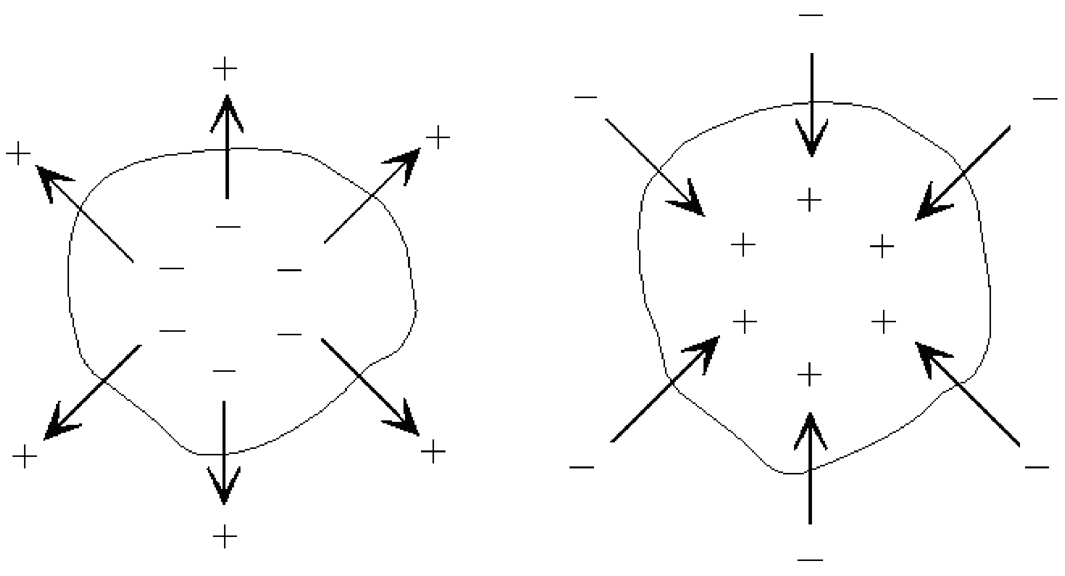


The charge $\boldsymbol{q}$ will still be the same

$$\therefore \sigma_b = \frac{q}{A_{\text{end}}} = \frac{q}{A}\cos\theta = \mathbf{P}\cdot\hat{\mathbf{n}}$$

Volume bound charges:

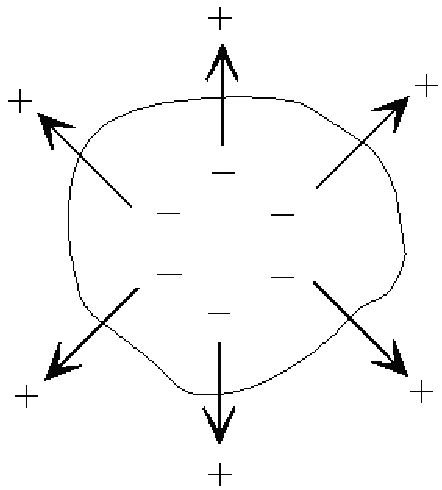
If there are **‘sources’** or **‘sinks’** of $\mathbf{P}$, there will be negative and positive charges accumulated in a small volume element enclosing the point:



$$\nabla \cdot \mathbf{P} > 0 \qquad\qquad \nabla \cdot \mathbf{P} < 0$$

Volume bound charges:

For example, when $\nabla \cdot \mathbf{P} > 0$, there will be some positive charges on the surface of the sphere, leaving equal amount of negative charges accumulated inside.



The amount of charge on the surface is

$$Q = \oint_S \mathbf{P} \cdot d\mathbf{a}$$

By definition,

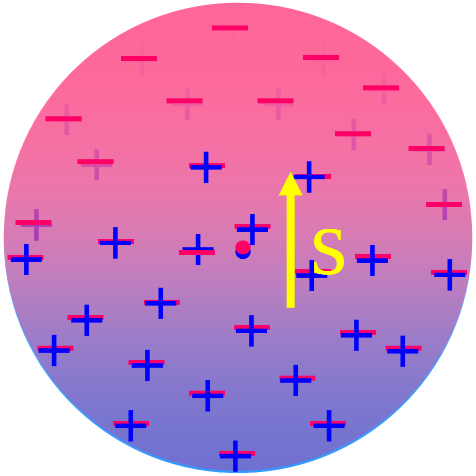
$$\rho_b = \lim_{\mathcal{V} \to 0} \frac{Q}{\mathcal{V}} = -\lim_{\mathcal{V} \to 0} \frac{1}{\mathcal{V}} \oint_S \mathbf{P} \cdot d\mathbf{a} = -\nabla \cdot \mathbf{P}$$

You can check that this is also true when $\nabla \cdot \mathbf{P} < 0$

Example:

A physical way to look at the last example is to consider the uniformly polarized sphere as two uniformly charged sphere displaced by a distance **s**.

Let the charge carried by the two spheres be q and –q, with density ρ *and* $-\rho$, *respectively. Then*



$$\mathbf{P} = \rho\mathbf{s} \quad \text{and} \quad \mathbf{p} = \frac{4}{3}\pi R^3 \mathbf{P} = q\mathbf{s}$$

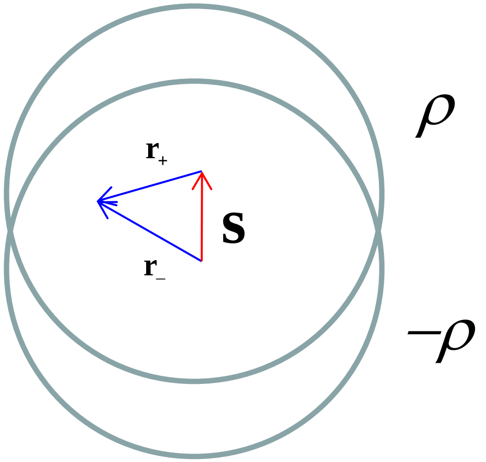
The E field due to a uniformly charged sphere is

$$E \times 4\pi r^2 = \frac{1}{\varepsilon_0} \times \rho \frac{4}{3}\pi r^3 \Rightarrow \mathbf{E} = \frac{\rho}{3\varepsilon_0}\mathbf{r}$$

where **r** *is the vector pointing from the center to the observation point*

Hence for two overlapping spheres with charge densities ρ and $-\rho$, in the overlapping region

$$\mathbf{E} = \frac{\rho}{3\varepsilon_0}\mathbf{r}_+ - \frac{\rho}{3\varepsilon_0}\mathbf{r}_- = \frac{\rho}{3\varepsilon_0}\left(\mathbf{r}_+ - \mathbf{r}_-\right) = -\frac{\rho}{3\varepsilon_0}\mathbf{s}$$

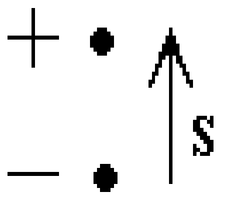


$$\mathbf{E} = -\frac{1}{3\varepsilon_0}\mathbf{P}$$

Outside the spheres, the field due to the positively charged sphere is the same as that due to a point charge $q = \rho \cdot \frac{4}{3}\pi R^3$ *at the center. And similarly for the field due to the negatively charged sphere.*

Therefore, the field outside is the same as that due to a dipole $\mathbf{p} = q\mathbf{s}$*:*

$$V = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$



Electric Fields in Matter

The Electric Displacement

Consider Gauss's Law in the presence of a dielectric:

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} \rho_{\text{total}}$$

$$= \frac{1}{\varepsilon_0} \left(\rho_f + \rho_b \right)$$

ρ_{total} : Total charge density

ρ_f : Free charge density

ρ_b : Bound charge density

$$\because \rho_b = -\nabla \cdot \mathbf{P}$$

$$\therefore \ \varepsilon_0 \nabla \cdot \mathbf{E} = \rho_f - \nabla \cdot \mathbf{P}$$

$$\Rightarrow \ \nabla \cdot \underbrace{\left(\varepsilon_0 \mathbf{E} + \mathbf{P}\right)} = \rho_f \qquad \boxed{\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}}$$

electric displacement D

Then,

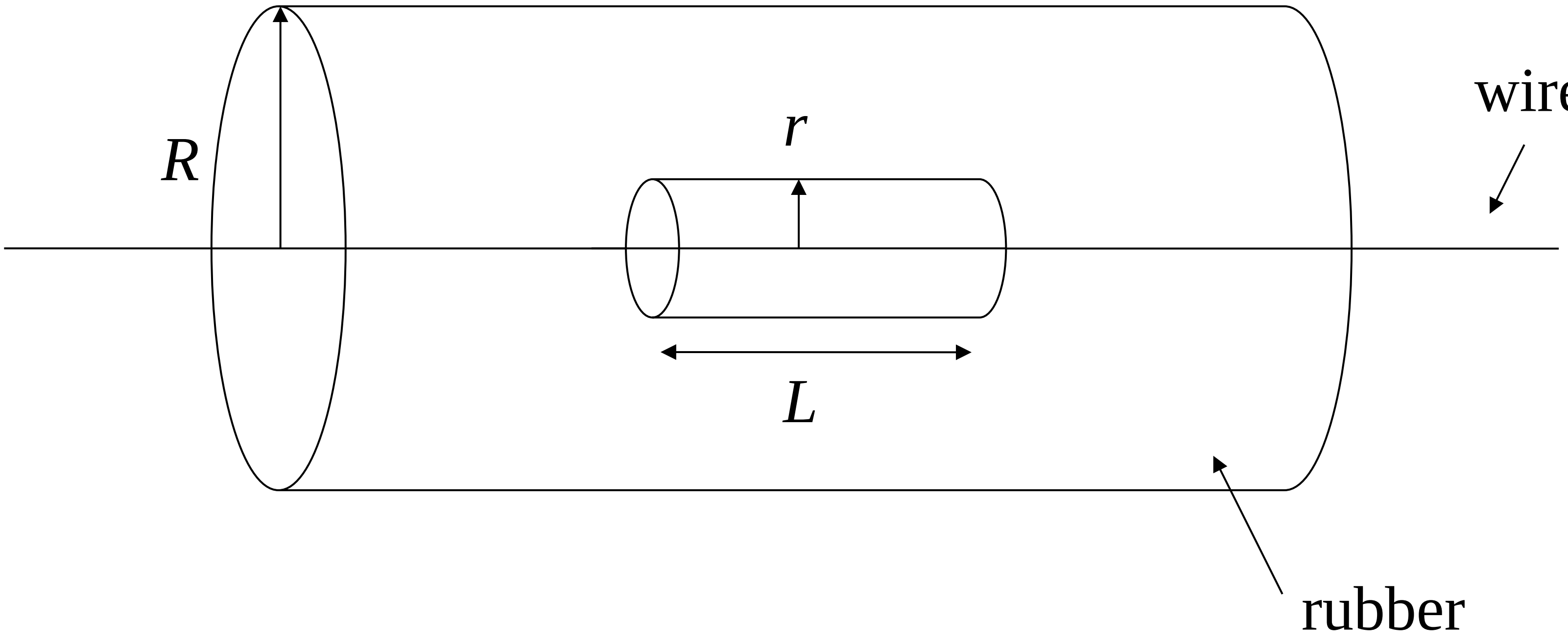
$$\nabla \cdot \mathbf{D} = \rho_f$$

Integral form:

$$\oint_{\text{surface}} \mathbf{D} \cdot d\mathbf{a} = Q_{f\text{enc}}$$

Example:

A long straight wire, carrying uniform line charge density λ , is surrounded by rubber insulation out to a radius *R*. Find the electric displacement.



Solution:

Since

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f\text{enc}}$$

$$D\left(2\pi rl\right) = \lambda l$$

$$\mathbf{D} = \frac{\lambda}{2\pi r}\hat{\mathbf{r}}$$

This is true both for $r \geq R$ and $r \leq R$

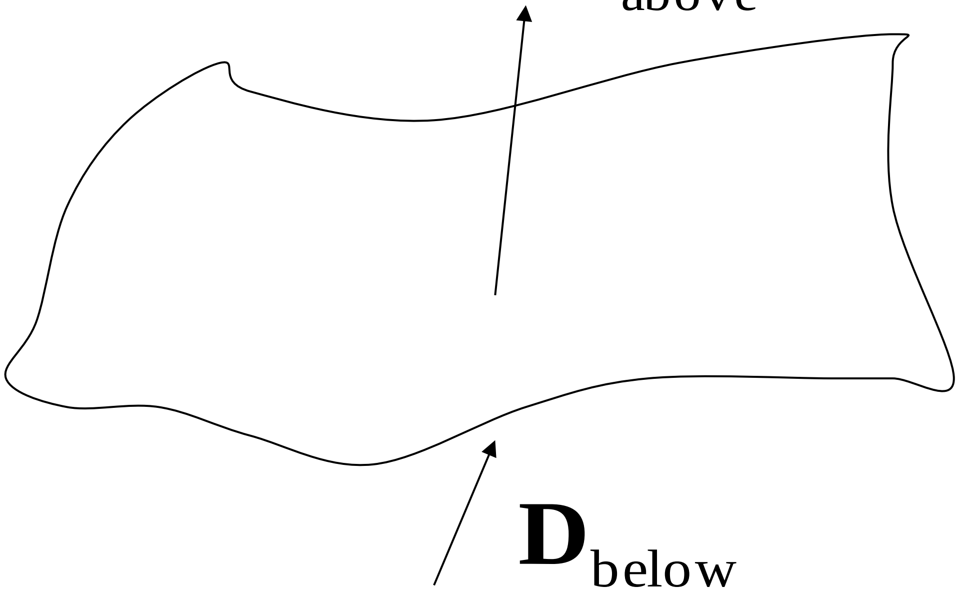
Outside the rubber, $\mathbf{P} = \mathbf{0}$

$$\therefore \ \mathbf{E} = \frac{1}{\varepsilon_0}\mathbf{D} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{\mathbf{r}}$$

Inside the rubber, $\mathbf{E}$ is unknown. Have to know $\mathbf{P}$ first.

Boundary Conditions of D:

Consider a surface with surface charge density σ



The surface charge may be due to bound charges and free charges: $$\sigma = \sigma_f + \sigma_b$$

Since $$\oint \mathbf{D}\cdot da = Q_{f\text{enc}}$$

$$\left(\mathbf{D}_{\text{above}} - \mathbf{D}_{\text{below}}\right)\cdot\hat{\mathbf{n}} = \sigma_f$$

i.e. $$D^{\perp}{}_{\text{above}} - D^{\perp}{}_{\text{below}} = \sigma_f$$

where $\hat{\mathbf{n}}$ is a unit normal vector pointing from “below” to “above”.

(cf. $\left(\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}}\right)\cdot\hat{\mathbf{n}} = \dfrac{\sigma}{\varepsilon_0}$)

For the parallel components, since

$$\nabla \times \mathbf{D} = \nabla \times (\varepsilon_0 \mathbf{E} + \mathbf{P})$$

$$= \varepsilon_0 \nabla \times \mathbf{E} + \nabla \times \mathbf{P}$$

$$= \nabla \times \mathbf{P}$$

$$\therefore \mathbf{D}^{//}_{\text{above}} - \mathbf{D}^{//}_{\text{below}} = \mathbf{P}^{//}_{\text{above}} - \mathbf{P}^{//}_{\text{below}}$$

(cf. $\mathbf{E}^{//}_{\text{above}} = \mathbf{E}^{//}_{\text{below}}$)

Since $\nabla \times \mathbf{P} \neq \mathbf{0}$ in general, so $\mathbf{D}$ is not curl free in general, and there is no "potential" for $\mathbf{D}$.

Although $\nabla \cdot \mathbf{D} = \rho_f$ is similar to $\nabla \cdot \mathbf{E} = \dfrac{\rho}{\varepsilon_0}$ there is no **"Coulomb's law"** for $\mathbf{D}$

Linear Dielectrics:

In many situations, the polarization is proportional to the E-field under the weak-field condition. Define

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$$

electric susceptibility

Materials obeying this relation are called ***linear dielectrics***.

Hence,

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

$$= \varepsilon_0 \mathbf{E} + \varepsilon_0 \chi_e \mathbf{E}$$

$$= \varepsilon_0 (1 + \chi_e) \mathbf{E}$$

Define,

Permittivity of free space

$$\varepsilon = \varepsilon_0 (1 + \chi_e)$$

$$\mathbf{D} = \varepsilon \mathbf{E}$$

Permittivity of the material

Dielectric constant: $K = \dfrac{\varepsilon}{\varepsilon_0} = 1 + \chi_e$

For linear dielectrics,

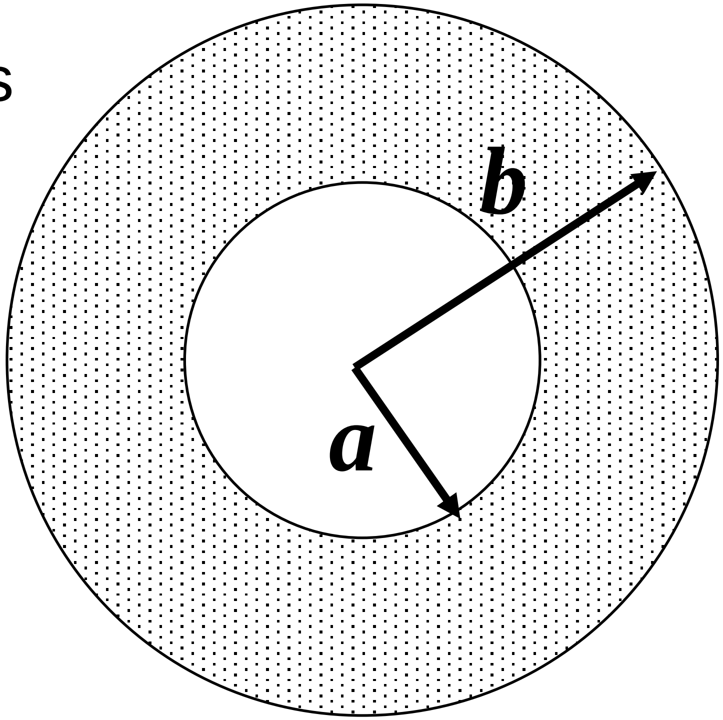
$$\rho_b = -\nabla \cdot \mathbf{P} = -\nabla \cdot \left(\varepsilon_0 \chi_e \mathbf{E}\right) = -\nabla \cdot \left(\varepsilon_0 \frac{\chi_e}{\varepsilon} \mathbf{D} \right) = -\left(\frac{\chi_e}{1 + \chi_e} \right) \rho_f$$

Therefore, if $\rho_f = 0 \quad \Longrightarrow \rho_b = 0$

In other words, all bound charges are on the surface if $\rho_f = 0$

Example:

A metal sphere of radius a carries a charge Q. It is surrounded, out to radius b, by linear dielectric material of permittivity ε .



Find the potential at the center (relative to infinity).

Solution:

By Gauss's law,

$$\mathbf{D} = \frac{Q}{4\pi r^2}\hat{\mathbf{r}} \quad , \qquad \text{for } r > a$$

$$\mathbf{E} = \begin{cases} \dfrac{Q}{4\pi\varepsilon r^2}\hat{\mathbf{r}} & \text{for } a < r < b \\ \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} & \text{for } r > b \end{cases}$$

Solution:

The potential at the center

$$V=-\int_{\infty}^{0}\mathbf{E}\cdot d\mathbf{l}=-\int_{\infty}^{b}\frac{Q}{4\pi\varepsilon_0 r^2}dr-\int_{b}^{a}\frac{Q}{4\pi\varepsilon r^2}dr-\int_{a}^{0}(0)\,dr$$

$$=\frac{Q}{4\pi}\left(\frac{1}{\varepsilon_0 b}+\frac{1}{\varepsilon a}-\frac{1}{\varepsilon b}\right)$$

Solution:

The polarization is:

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E} = \frac{\varepsilon_0 \chi_e Q}{4\pi\varepsilon r^2}\hat{\mathbf{r}}$$

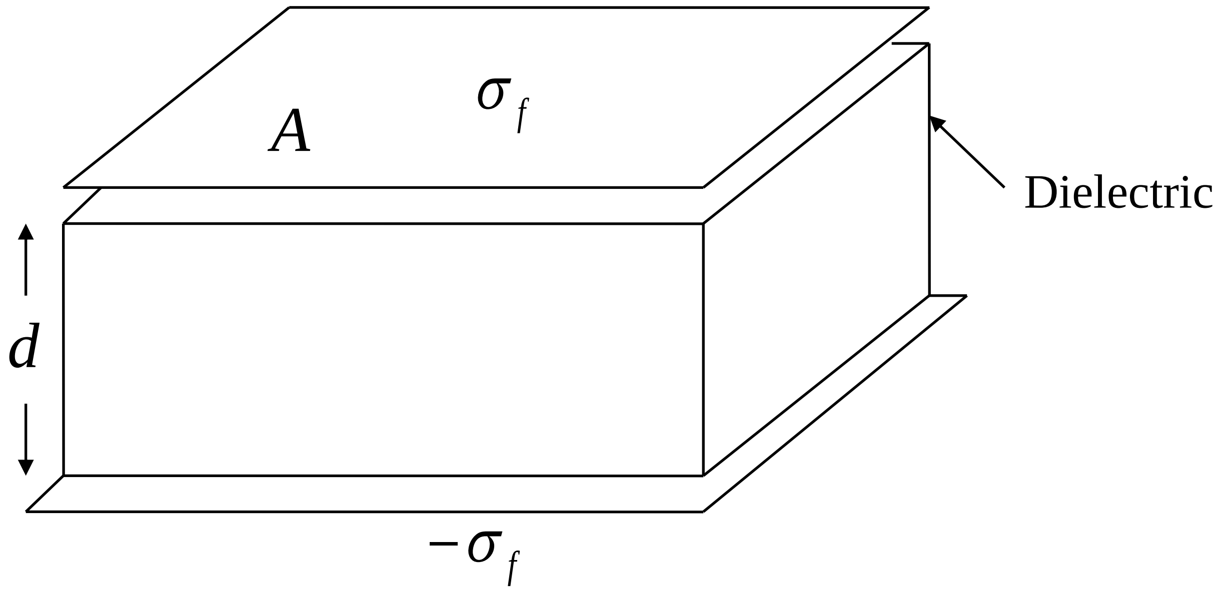
$$\therefore \quad \rho_b = -\nabla \cdot \mathbf{P} = 0$$

and

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = \begin{cases} \dfrac{\varepsilon_0 \chi_e Q}{4\pi\varepsilon b^2} & \text{for } r = b \\ -\dfrac{\varepsilon_0 \chi_e Q}{4\pi\varepsilon a^2} & \text{for } r = a \end{cases}$$

Example:

A parallel-plate capacitor is filled with insulating material of dielectric constant *K*. What effect does this have on its capacitance?



Solution:

From Gauss's law,

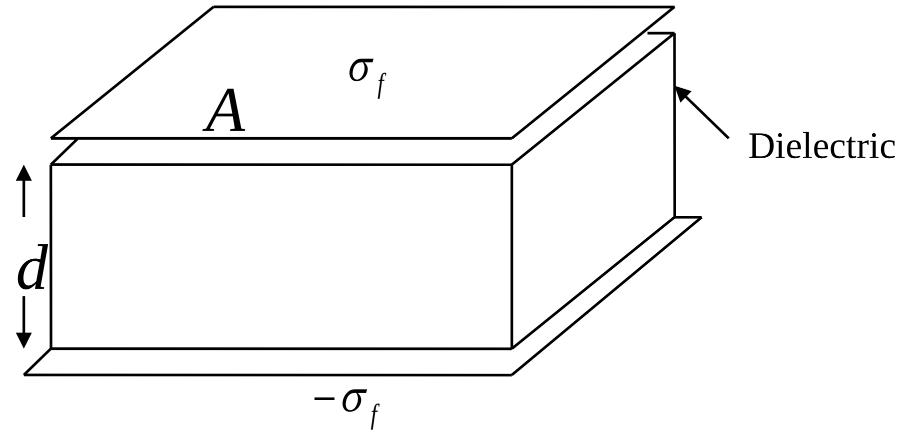
$$D = \sigma_f$$

$$E = \frac{D}{\varepsilon} = \frac{\sigma_f}{\varepsilon}$$

$$V = Ed = \frac{\sigma_f d}{\varepsilon}$$

$$= Q\frac{d}{\varepsilon A}$$

$$\therefore C = \frac{\varepsilon A}{d}$$



In vacuum,

$$C_{\text{vac}} = \frac{\varepsilon_0 A}{d}$$

$$\therefore C = \frac{\varepsilon}{\varepsilon_0} C_{\text{vac}} = K C_{\text{vac}}$$

Boundary Conditions in Linear Dielectrics:

Consider the interface of two dielectric media

If $\sigma_f = 0$, the normal component of $\mathbf{D}$ is continuous:

$\mathbf{D}_1$

$$\mathbf{D}_1^{\perp} = \mathbf{D}_2^{\perp}$$

ε_1

$$\therefore \varepsilon_1 \mathbf{E}_1^{\perp} = \varepsilon_2 \mathbf{E}_2^{\perp}$$

ε_2

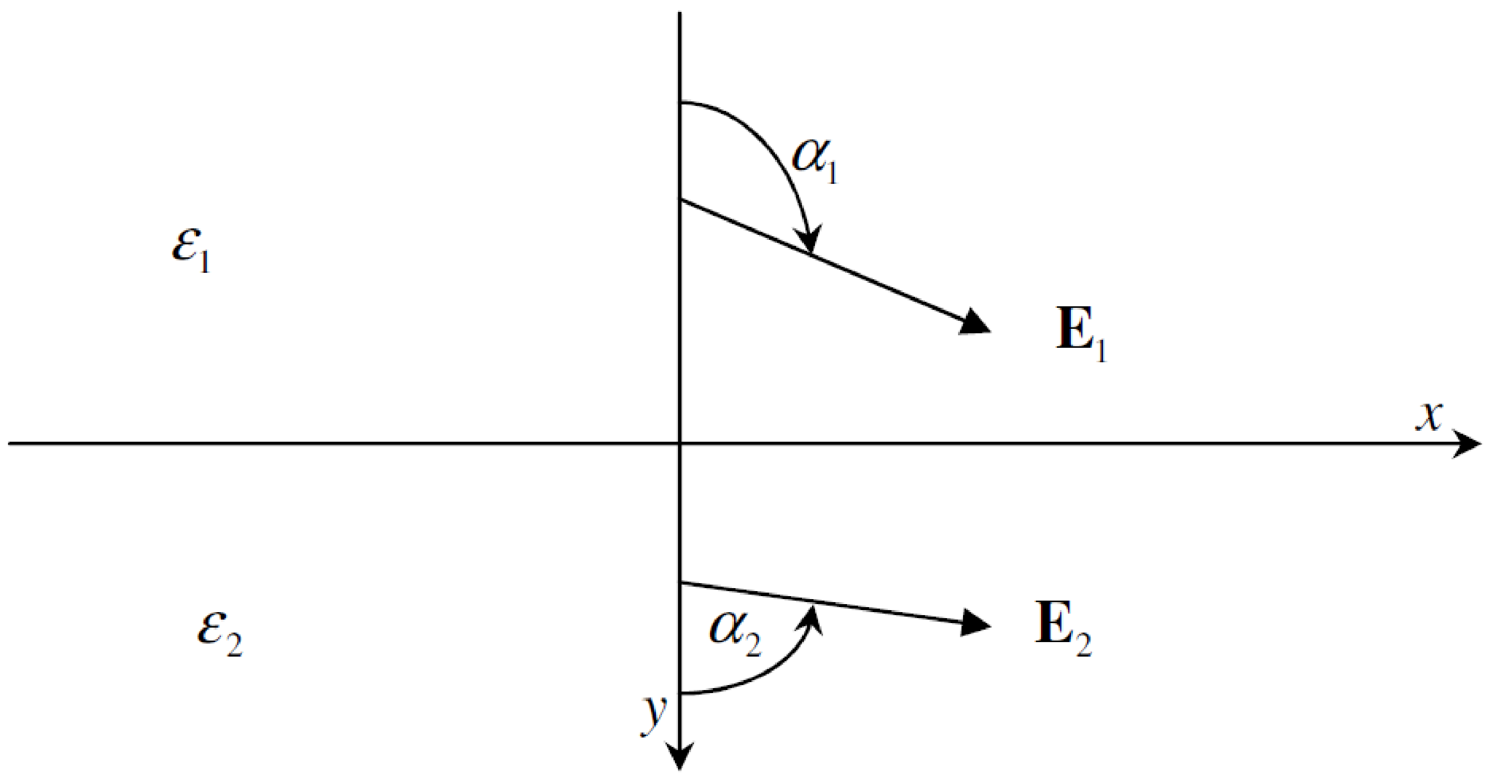
On the other hand, since $\nabla \times \mathbf{E} = \mathbf{0}$

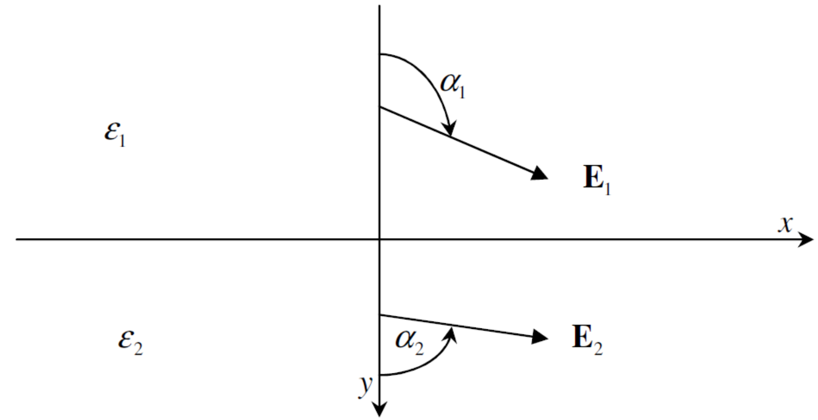
$\mathbf{D}_2$

$$\therefore \mathbf{E}_1^{//} = \mathbf{E}_2^{//}$$

$$\therefore \frac{1}{\varepsilon_1}\mathbf{D}_1^{//} = \frac{1}{\varepsilon_2}\mathbf{D}_2^{//}$$

At the interface between two dielectrics, with permittivities ε_1 and ε_2, the electric field in the first dielectric has magnitude E_1 and makes an angle α_1 with the normal of the interface. What is the magnitude and direction of the electric field in the second dielectric?



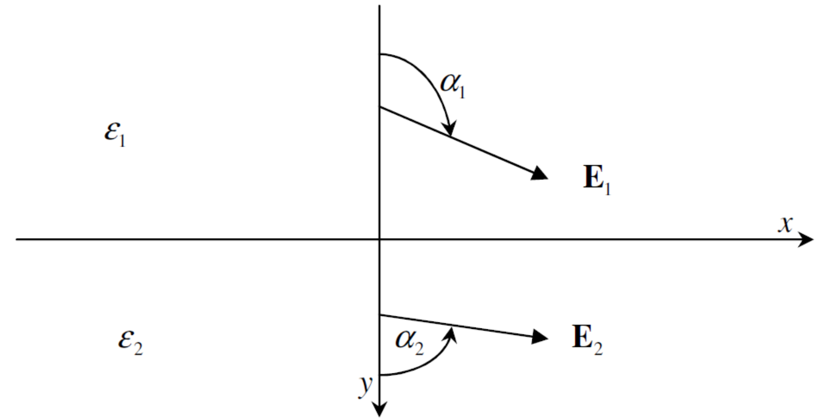


$$\because \mathbf{E}^{\parallel}{}_{\text{above}} = \mathbf{E}^{\parallel}{}_{\text{below}} \Rightarrow E_{1x} = E_{2x} \; --- (1)$$

$$\because D^{\perp}{}_{\text{above}} - D^{\perp}{}_{\text{below}} = \sigma_f = 0 \Rightarrow D_{1y} = D_{2y} \; --- (2)$$

$$\tan(\pi - \alpha_1) = \frac{E_{1x}}{E_{1y}} = \frac{\varepsilon_1 E_{1x}}{D_{1y}} --- (3)$$

$$\tan \alpha_2 = \frac{E_{2x}}{E_{2y}} = \frac{\varepsilon_2 E_{2x}}{D_{2y}} --- (4)$$

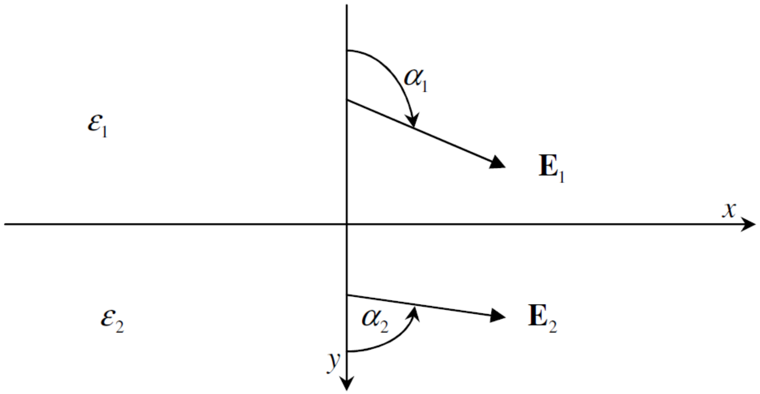


From (1) and (2), $\dfrac{E_{1x}}{D_{1y}} = \dfrac{E_{2x}}{D_{2y}}$

and (3)/(4) ==> $\dfrac{\tan(\pi-\alpha_1)}{\tan\alpha_2} = \dfrac{\varepsilon_1 E_{1x} / D_{1y}}{\varepsilon_2 E_{2x} / D_{2y}} = \dfrac{\varepsilon_1}{\varepsilon_2} \Rightarrow \dfrac{\tan\alpha_1}{\tan\alpha_2} = -\dfrac{\varepsilon_1}{\varepsilon_2}$

$$\therefore \alpha_2 = \tan^{-1}\left(-\frac{\varepsilon_2}{\varepsilon_1}\tan\alpha_1\right)$$

To determine the magnitude of the electric field,



$$\left|\mathbf{E}_1\right|^2 = E_{1x}{}^2 + E_{1y}{}^2 = E_{1x}{}^2 + \frac{D_{1y}{}^2}{\varepsilon_1{}^2} \Rightarrow \varepsilon_1{}^2\left|\mathbf{E}_1\right|^2 = \varepsilon_1{}^2 E_{1x}{}^2 + D_{1y}{}^2 \quad ---(5)$$

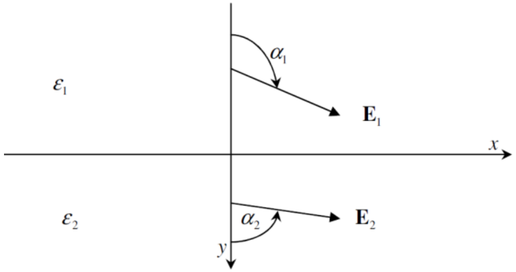
Similarly, $\varepsilon_2{}^2\left|\mathbf{E}_2\right|^2 = \varepsilon_2{}^2 E_{2x}{}^2 + D_{2y}{}^2 \quad ---(6)$

(5)-(6):

$$\varepsilon_1{}^2\left|\mathbf{E}_1\right|^2 - \varepsilon_2{}^2\left|\mathbf{E}_2\right|^2 = \varepsilon_1{}^2 E_{1x}{}^2 - \varepsilon_2^2 E_{2x}{}^2$$

$$= (\varepsilon_1{}^2 - \varepsilon_2{}^2) E_{1x}{}^2$$

$$= (\varepsilon_1{}^2 - \varepsilon_2{}^2)\left|\mathbf{E}_1\right|^2 \sin^2(\pi - \alpha_1)$$



$$\Rightarrow \varepsilon_2{}^2 |\mathbf{E}_2|^2 = \varepsilon_1{}^2 |\mathbf{E}_1|^2 - (\varepsilon_1{}^2 - \varepsilon_2{}^2)|\mathbf{E}_1|^2 \sin^2\alpha_1$$

$$\Rightarrow |\mathbf{E}_2|^2 = |\mathbf{E}_1|^2 \left(\frac{\varepsilon_1{}^2}{\varepsilon_2{}^2} - \frac{(\varepsilon_1{}^2 - \varepsilon_2{}^2)\sin^2\alpha_1}{\varepsilon_2{}^2} \right)$$

$$= |\mathbf{E}_1|^2 \left(\frac{\varepsilon_1{}^2(1-\sin^2\alpha_1) + \varepsilon_2{}^2 \sin^2\alpha_1}{\varepsilon_2{}^2} \right)$$

$$= |\mathbf{E}_1|^2 \left(\frac{\varepsilon_1{}^2 \cos^2\alpha_1 + \varepsilon_2{}^2 \sin^2\alpha_1}{\varepsilon_2{}^2} \right)$$

Note: when $\alpha_1 = \pi/2$, i.e. along the boundary, $|\mathbf{E}_2|^2 = |\mathbf{E}_1|^2$.

when $\alpha_1 = \pi$, i.e. perpendicular to the boundary, $\varepsilon_1^2 |\mathbf{E}_1|^2 = \varepsilon_2^2 |\mathbf{E}_2|^2$

Energy in Dielectric Systems:

Consider the energy required to construct the system by moving free-charges to their final position, a bit at a time, and allowing the dielectrics to respond accordingly.

$$\Delta W = \int (\Delta \rho_f) V d\tau$$

Since $\nabla \cdot \mathbf{D} = \rho_f \quad , \quad \Delta \rho_f = \nabla \cdot (\Delta \mathbf{D})$

$$\therefore \Delta W = \int \nabla \cdot (\Delta \mathbf{D}) V d\tau$$

$$= \int \left[\nabla \cdot (V \Delta \mathbf{D}) - (\nabla V) \cdot \Delta \mathbf{D} \right] d\tau$$

$$= \int \left[\nabla \cdot (V \Delta \mathbf{D}) + \mathbf{E} \cdot \Delta \mathbf{D} \right] d\tau$$

$$\int \nabla \cdot (V\Delta\mathbf{D})\,d\tau = \int_S V\Delta\mathbf{D} \cdot d\mathbf{a} = 0$$

if the volume integral includes the entire space.

$$\therefore \Delta W = \int \mathbf{E} \cdot \Delta\mathbf{D}\,d\tau$$

For linear dielectrics, $\mathbf{D} = \varepsilon\mathbf{E}$

$$\Delta W = \varepsilon \int \mathbf{E} \cdot \Delta\mathbf{E}\,d\tau$$

$$= \Delta\left[\frac{1}{2}\varepsilon \int E^2\,d\tau\right]$$

For an infinitesimal ΔE!

$$\therefore W = \frac{1}{2}\varepsilon \int \mathbf{E} \cdot \mathbf{E}\,d\tau = \frac{1}{2}\int \mathbf{D} \cdot \mathbf{E}\,d\tau$$