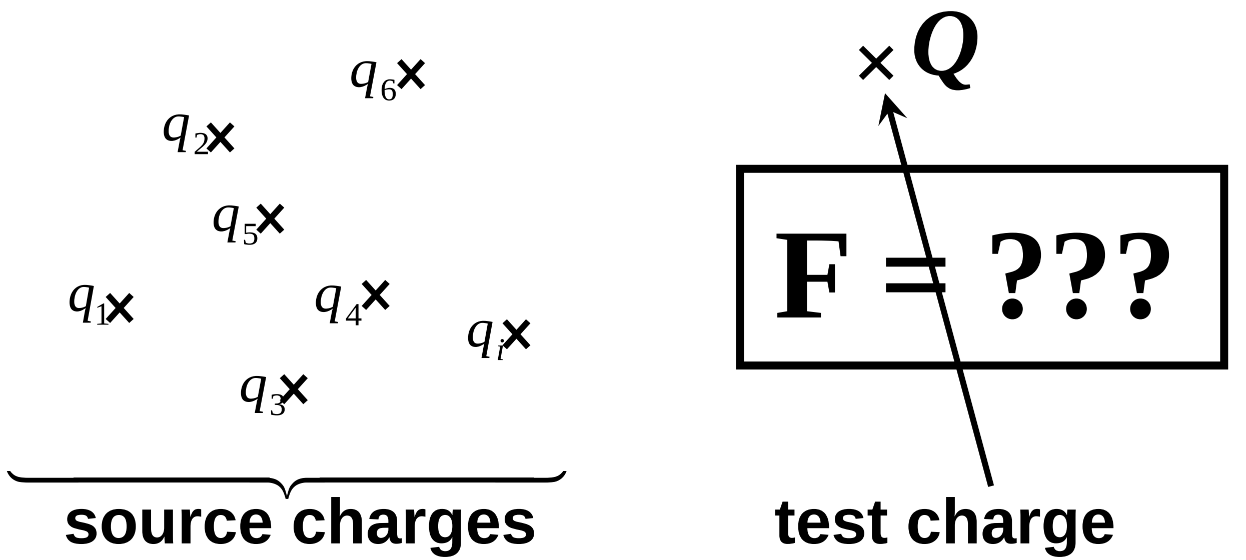
Electrostatics

What is the force exerted on a test charge Q, by some source charges $q_1, q_2, q_3, \cdots$?



Principle of superposition:

- The interaction between any two charges is completely unaffected by the presence of others.

- In general, the force depends not only on the position of q, but also its velocity and acceleration.

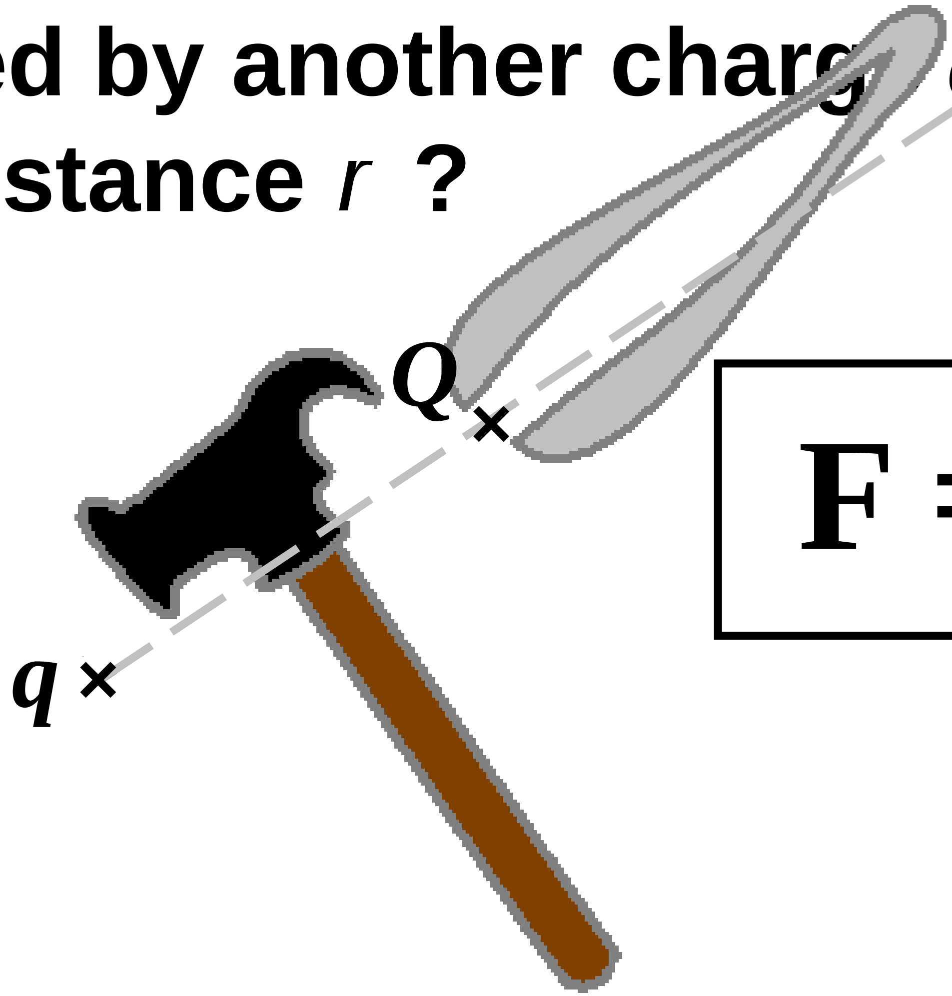
Electrostatics:

- The source charges are ***stationary***.

Electrostatics

Coulomb’s Law

What is the force on a test charge Q exerted by another charge q at rest at a distance r ?



F = ???

Experimental facts:

- The direction of the force is along the line joining the two charges, being either repulsive or attractive.

- The charges can be divided into two separate groups, called positive (plus) and negative (minus).

- Like charges repel each other while the forces between unlike charges are attractive.

Experimental facts:

- The magnitude of the force is proportional to the product of the charges (principle of superposition) and inversely proportional to the square of distance.

$$\mathbf{F} \propto \frac{qQ}{r^2}\hat{r}$$

Units:

Gaussian (cgs): $$\mathbf{F} = \frac{qQ}{r^2}\hat{r}$$

SI (mks): $$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}\hat{r}$$

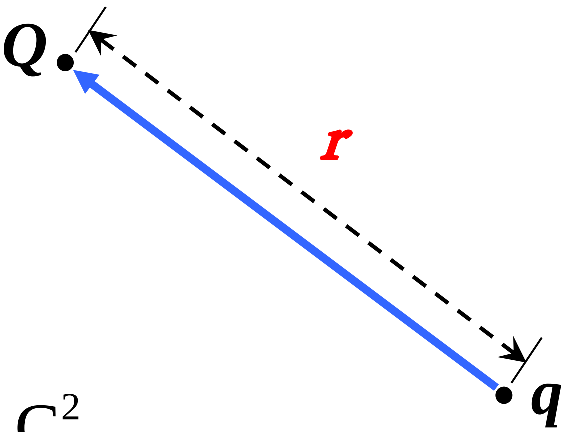
We shall use SI units.

Unit of charge: Coulomb (C)

Definition: The amount of charges flowing through a wire in 1 second when the current is 1 Ampere (A), which will be defined later.

Coulomb's Law:

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}\hat{r}$$



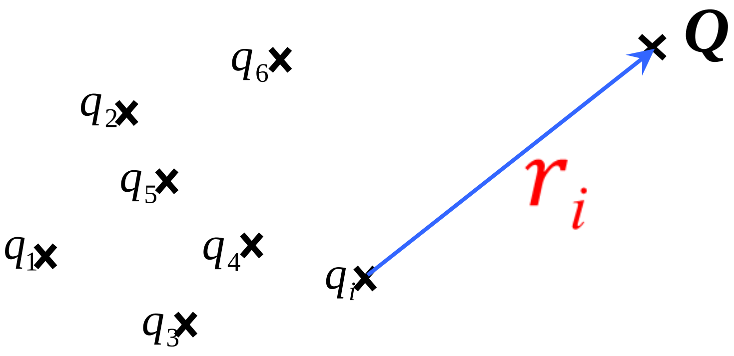
Permittivity of free space

$$\varepsilon_0 = 8.85\times10^{-12}\ \frac{\mathrm{C}^2}{\mathrm{N}\cdot\mathrm{m}^2}$$

Electric Field:

$$\mathbf{E} = \frac{\mathbf{F}}{Q} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

From the principle of superposition, for a set of point charges $q_1, q_2, \cdots q_n$

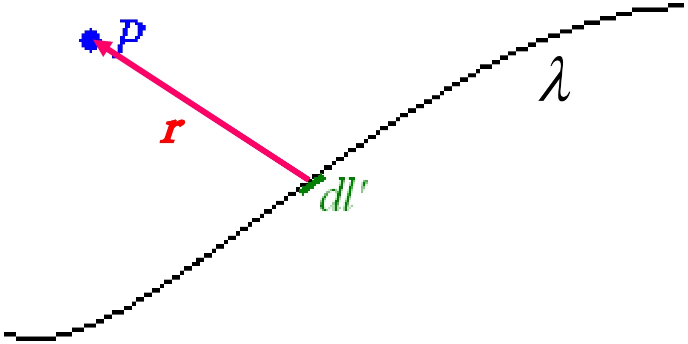


$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^2}\hat{r}_i$$

Continuous charge distributions:

- Charges along a line with linear charge density $\lambda(\mathbf{r}')$

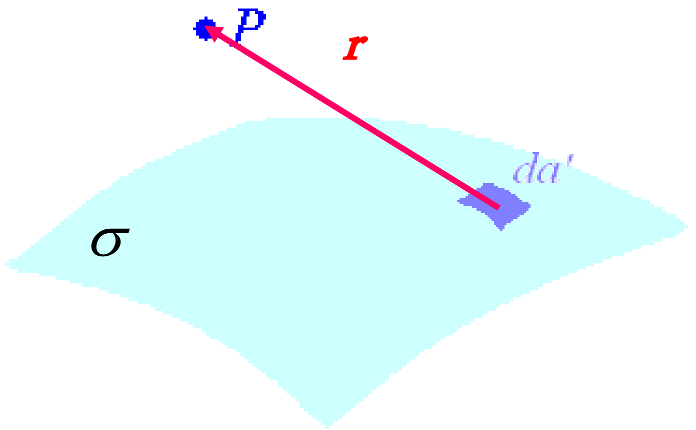
$$dq = \lambda dl' \rightarrow \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_P \frac{\lambda(\mathbf{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}dl'$$



Continuous charge distributions:

- Charges on a surface with surface charge density $\sigma(\mathbf{r}')$

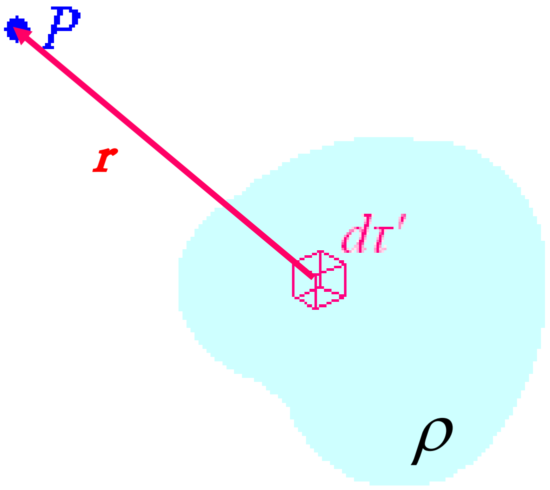
$$dq = \sigma da' \rightarrow \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\sigma(\mathbf{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\, da'$$



Continuous charge distributions:

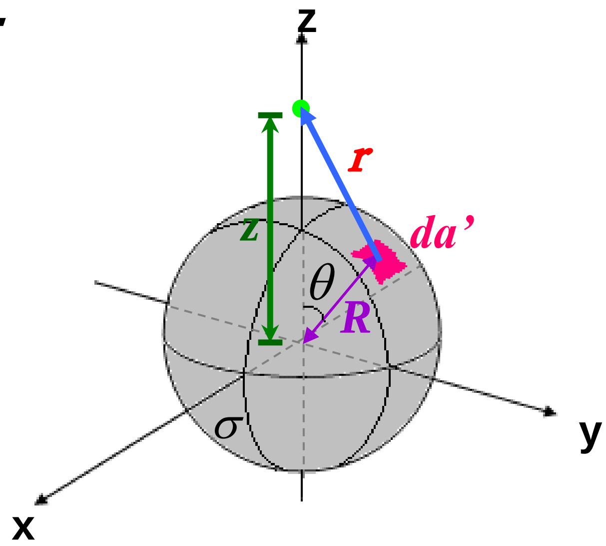
- Charges fill a volume with volume charge density $\rho(\mathbf{r}')$

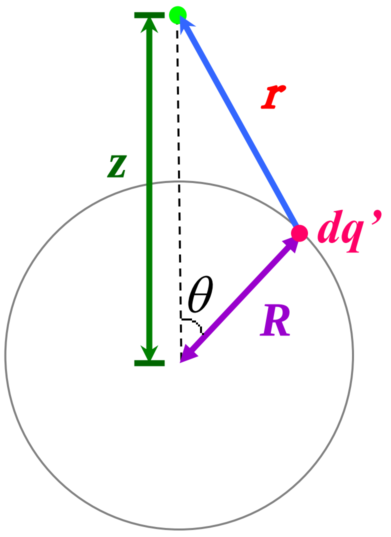
$$dq = \rho d\tau' \rightarrow \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\, d\tau'$$



Example:

Find the electric field a distance *z* from the center of a spherical surface of radius *R*, which carries a uniform charge density σ





Ans:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\sigma(\mathbf{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\,da'$$

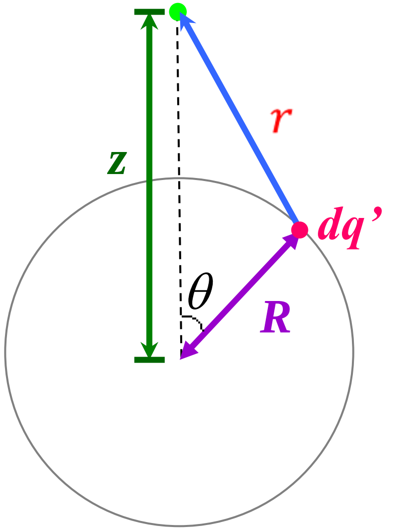
$$= \frac{\sigma}{4\pi\varepsilon_0}\int_S \frac{R^2 \sin\theta d\theta d\phi}{\mathscr{r}^3}\mathscr{r}$$

Without loss of generality, let's assume that the observation point is on the z-axis for simplicity. Hence

$$\mathscr{r} = z\hat{\mathbf{z}} - R\hat{\mathbf{r}}$$

and,

$$\mathscr{r}^2 = R^2 + z^2 - 2Rz\cos\theta$$



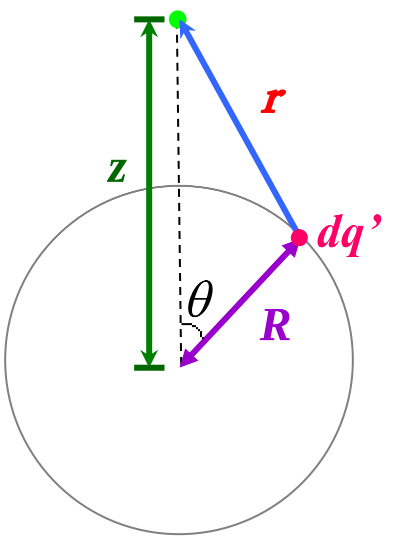
From symmetry, the E-field is along the z direction

$$E_z = \frac{\sigma}{4\pi\varepsilon_0}\int_S \frac{R^2 \sin\theta d\theta d\phi}{\mathscr{r}^3}\mathscr{r}\cdot\hat{\mathbf{z}}$$

$$= \frac{\sigma R^2}{4\pi\varepsilon_0}\int_S \frac{\sin\theta d\theta d\phi}{\left(R^2 + z^2 - 2Rz\cos\theta\right)^{3/2}}\left(z - R\cos\theta\right)$$

$$= \begin{cases} \dfrac{q}{4\pi\varepsilon_0 z^2} & \text{for } z > R \\ 0 & \text{for } z < R \end{cases}$$

(Refer to the supplementary notes for details)



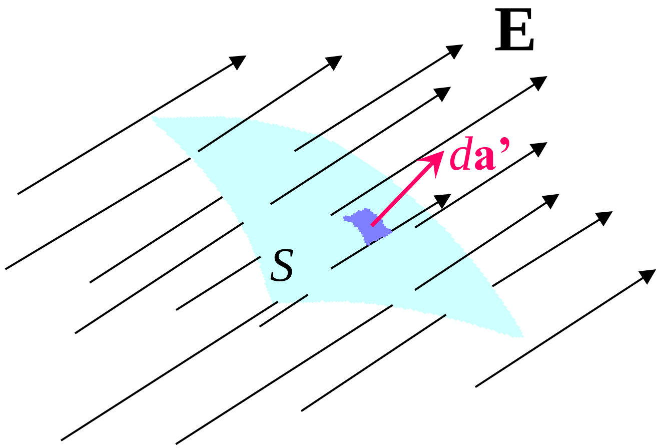
Electrostatics

Gauss’s Law

Gauss's Law in Integral Form

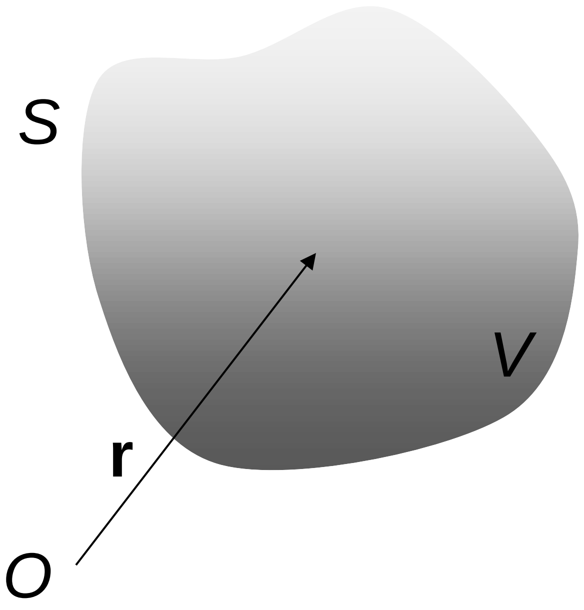
- ***Field lines: Connect the arrows of the* E *field vector***
- ***Flux of* E *through a surface* S*:***
  ***The surface integral of* E *over* S**

$$\Phi_E = \int_S \mathbf{E} \cdot d\mathbf{a}$$



Gauss's Law in Differential Form

Consider the divergence of E-field at a point r. To evaluate the divergence, one may consider a close surface S enclosing a small region V including r.



The divergence of E-field is given by

$$\nabla \cdot \mathbf{E} = \lim_{\mathcal{V}\to 0} \frac{1}{\mathcal{V}} \oint_S \mathbf{E} \cdot d\mathbf{a}$$

However, from Gauss's Law in integral form:

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\varepsilon_0}$$

Hence,

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} \lim_{\mathcal{V}\to 0} \frac{Q_{\text{enc}}}{\mathcal{V}}$$

The right hand side is just the volume charge density by definition. Hence,

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}}$$

which is the Gauss's law in differential form.

Notice that one can also obtain the same result by using Dirac Delta function:

From Coulomb's Law,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}} \frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\,d\tau'$$

If we take the divergence of both sides about r (not r'), we have

$$\nabla\cdot\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}} \rho(\mathbf{r}')\nabla\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)d\tau' = \frac{\rho(\mathbf{r})}{\varepsilon_0}$$

since

$$\nabla\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\mathbf{r}-\mathbf{r}') = 4\pi\delta^3(\mathscr{r})$$

Application of Gauss's Law

Gauss's Law (integral form) is extremely powerful in computing the electric field when the system exhibits some kind of symmetries, and with the suitable choice of Gaussian surfaces:

Spherical symmetry

Cylindrical symmetry

Plane symmetry

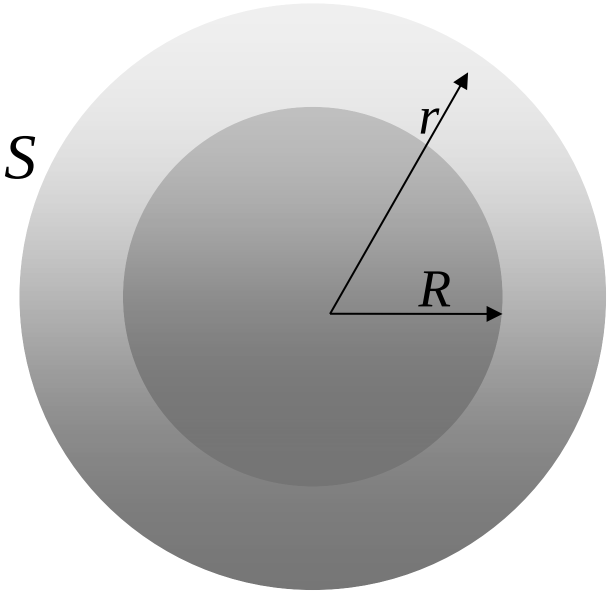
- **Spherical symmetry:** Spherical Gaussian surface concentric with the center of rotational symmetry.

- **Cylindrical symmetry:** Cylindrical Gaussian surface coaxial with the axis of rotational symmetry.

- **Plane symmetry:** "Pillbox" Gaussian surface bisected by the surface.

E field inside and outside a charged solid sphere with radius *R* and charge density $\rho(r)$, which depends on *r* only.

There is spherical symmetry in this problem. Consider a concentric Gaussian surface S with radius r.



Due to symmetry, the E field on the Gaussian surface is along the radial direction, i.e.,

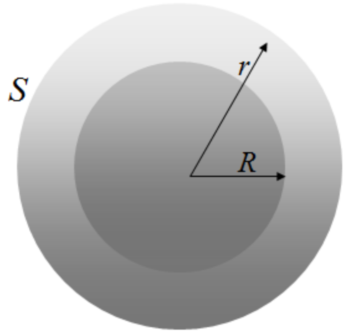
$$\mathbf{E} = E\hat{\mathbf{r}}$$

By Gauss's Law, if *r* > *R*,

$$\int_S \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\varepsilon_0} \int_{\text{Sphere}} \rho \, d\tau$$

$$E \times 4\pi r^2 = \frac{q}{\varepsilon_0}$$

$$\mathbf{E} = E\hat{\mathbf{r}} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}$$



The situation is exactly the same as if the total charge, *q*, were located at the center.

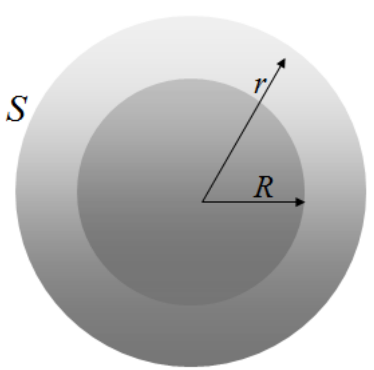
For *r* < *R*,

$$\int_S \mathbf{E}\cdot d\mathbf{a} = \frac{1}{\varepsilon_0}\int_{\mathcal{V}} \rho \, d\tau$$

where V is the volume enclosed by *S*. So

$$E \times 4\pi r^2 = \frac{1}{\varepsilon_0}\int_0^r \rho(r') 4\pi r'^2 dr'$$

$$\mathbf{E} = E\hat{\mathbf{r}} = \frac{1}{4\pi\varepsilon_0}\frac{\int_0^r \rho(r') 4\pi r'^2 dr'}{r^2}\hat{\mathbf{r}}$$



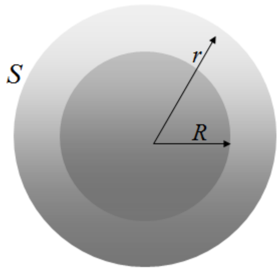
The situation is exactly the same as if the charge enclosed by *S* were located at the center and the charge outside *S* gave zero contribution.

In particular, for uniform charge distribution,

$$\int_0^r \rho(r') 4\pi r'^2 dr' = \rho \frac{4\pi r^3}{3} = \rho \frac{4\pi R^3}{3}\left(\frac{r}{R}\right)^3 = q\left(\frac{r}{R}\right)^3$$

Hence

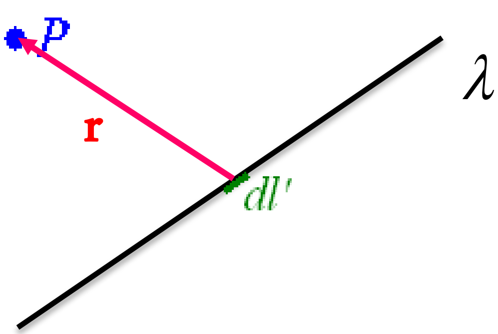
$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\left(\frac{r}{R}\right)^3 \hat{\mathbf{r}} = \frac{1}{4\pi\varepsilon_0}\frac{qr}{R^3}\hat{\mathbf{r}}$$

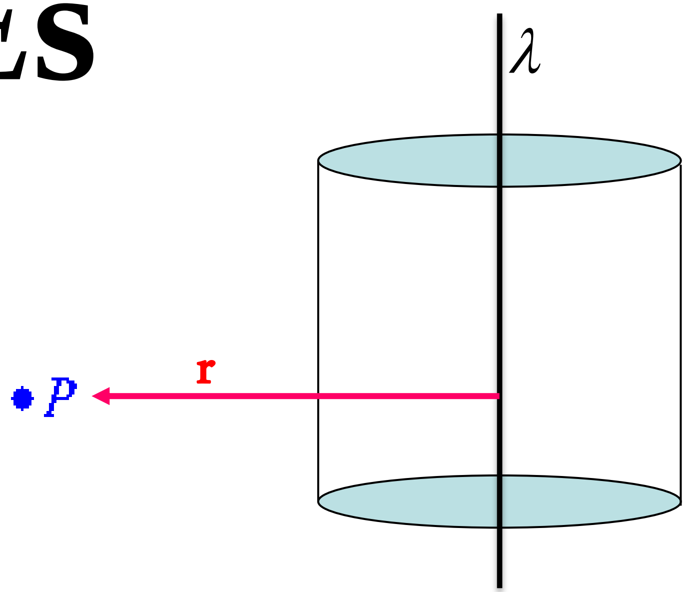


E field at a distance *s* from a line charge with linear charge density λ

There is cylindrical symmetry in this problem. Consider a cylindrical Gaussian surface coaxial with the axis of symmetry, with radius *s* and length *l*. By symmetry,

$$\mathbf{E} = E\hat{\mathbf{s}}$$

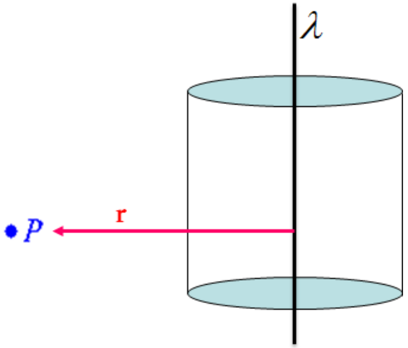




Therefore, one only has to evaluate the flux through the curved surface of the Gaussian cylinder.

$$E \times 2\pi s l = \frac{\lambda l}{\varepsilon_0}$$

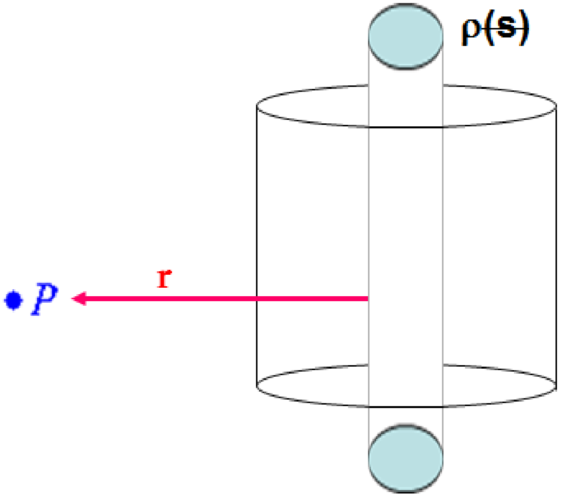
$$\boxed{\mathbf{E} = E\hat{\mathbf{s}} = \frac{\lambda}{2\pi\varepsilon_0 s}\hat{\mathbf{s}}}$$



E field inside and outside an infinite long cylinder with radius R carrying a charge density $\rho(s)$ which depends on s only.

There is cylindrical symmetry in this problem. Consider a cylindrical Gaussian surface coaxial with the axis of symmetry, with radius *s* and length *l*. By symmetry,

$$\mathbf{E} = E\hat{\mathbf{s}}$$

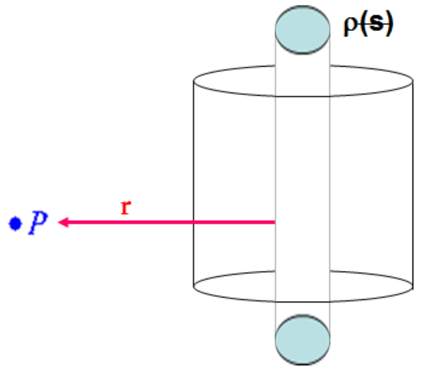


Therefore, one only has to evaluate the flux through the curved surface of the Gaussian cylinder.

If *s* > *R*,

$$E \times 2\pi sl = \frac{1}{\varepsilon_0} \int_0^R \rho(s') 2\pi l s' ds'$$

$$\mathbf{E} = E\hat{\mathbf{s}} = \frac{\int_0^R \rho(s') 2\pi s' ds'}{2\pi\varepsilon_0 s} \hat{\mathbf{s}}$$



i.e., it is the same as the case of a line charge with linear charge density

$$\lambda = \int_0^R \rho(s') 2\pi s' ds'$$

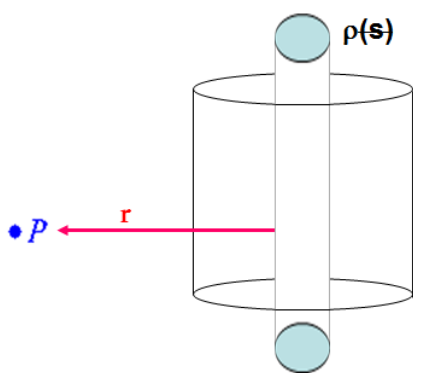
In particular, for constant ρ

$$\lambda = 2\pi\rho \int_0^R s' ds'$$

$$= \pi\rho R^2$$

and

$$\mathbf{E} = E\hat{\mathbf{s}} = \frac{\rho R^2}{2\varepsilon_0 s}\hat{\mathbf{s}}$$



If *s* < *R*,

$$E \times 2\pi s l = \frac{1}{\varepsilon_0} \int_0^s \rho(s') 2\pi l s' ds'$$

$$\mathbf{E} = E\hat{\mathbf{s}} = \frac{\int_0^s \rho(s') 2\pi s' ds'}{2\pi \varepsilon_0 s} \hat{\mathbf{s}}$$

i.e., it is as if the charges outside the Gaussian surface gave no contribution to the E field.

In particular, for constant ρ

$$\lambda = 2\pi\rho\int_0^s s' ds'$$

$$= \pi\rho s^2$$

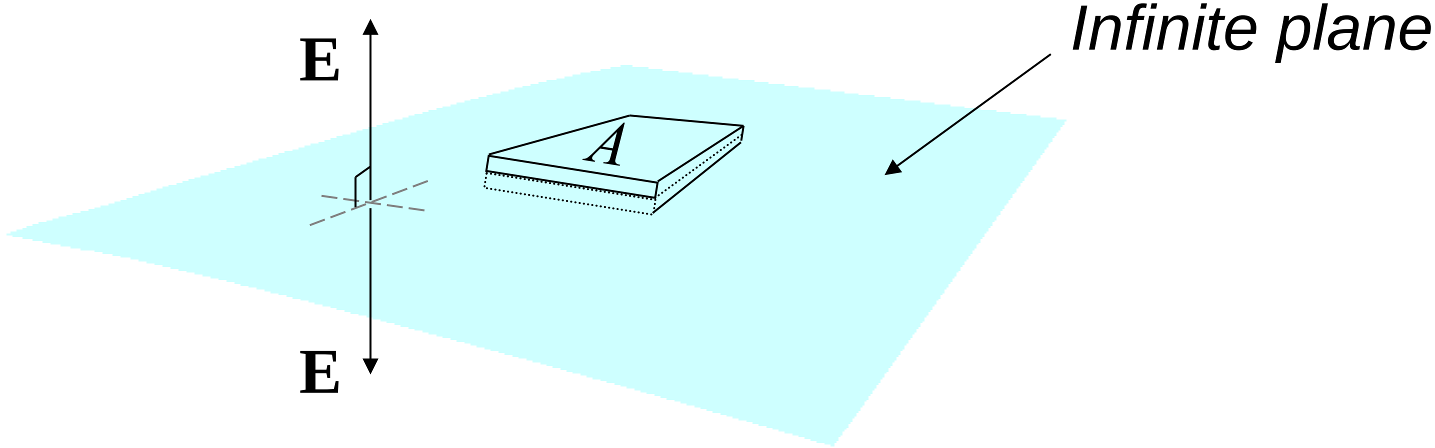
and

$$\mathbf{E} = E\hat{\mathbf{s}} = \frac{\rho s}{2\varepsilon_0}\hat{\mathbf{s}}$$

An infinite plane carrying a uniform surface charge density σ

There is plane symmetry in this problem.

Consider a "Gaussian pillbox" extending equal distances above and below the plane.
Let the area of the surface parallel to the plane be *A*.



Without loss of generality, let the plane be the *x*-*y* plane. Then by symmetry, the E field above the plane is

$$\mathbf{E} = E\hat{\mathbf{z}}$$

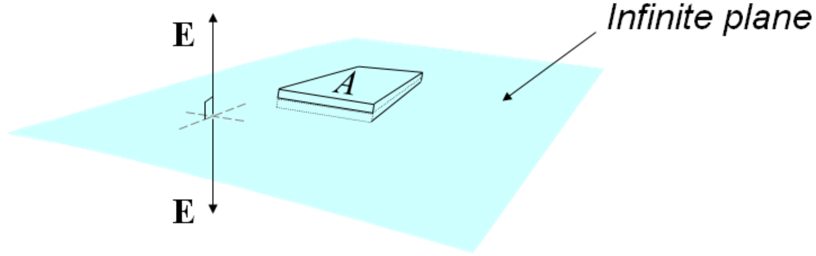
while that below the plane at an equal distance is

$$\mathbf{E} = -E\hat{\mathbf{z}}$$

Then by Gauss's law,

$$\Phi = 2EA = \frac{Q_{\text{enc}}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

$$\therefore E = \frac{\sigma}{2\varepsilon_0}$$



Above the *x*-*y* plane

$$\mathbf{E} = \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{z}}$$

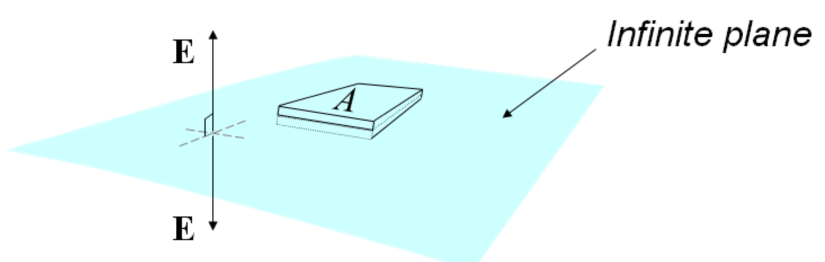
Below the *x*-*y* plane

$$\mathbf{E} = -\frac{\sigma}{2\varepsilon_0}\hat{\mathbf{z}}$$

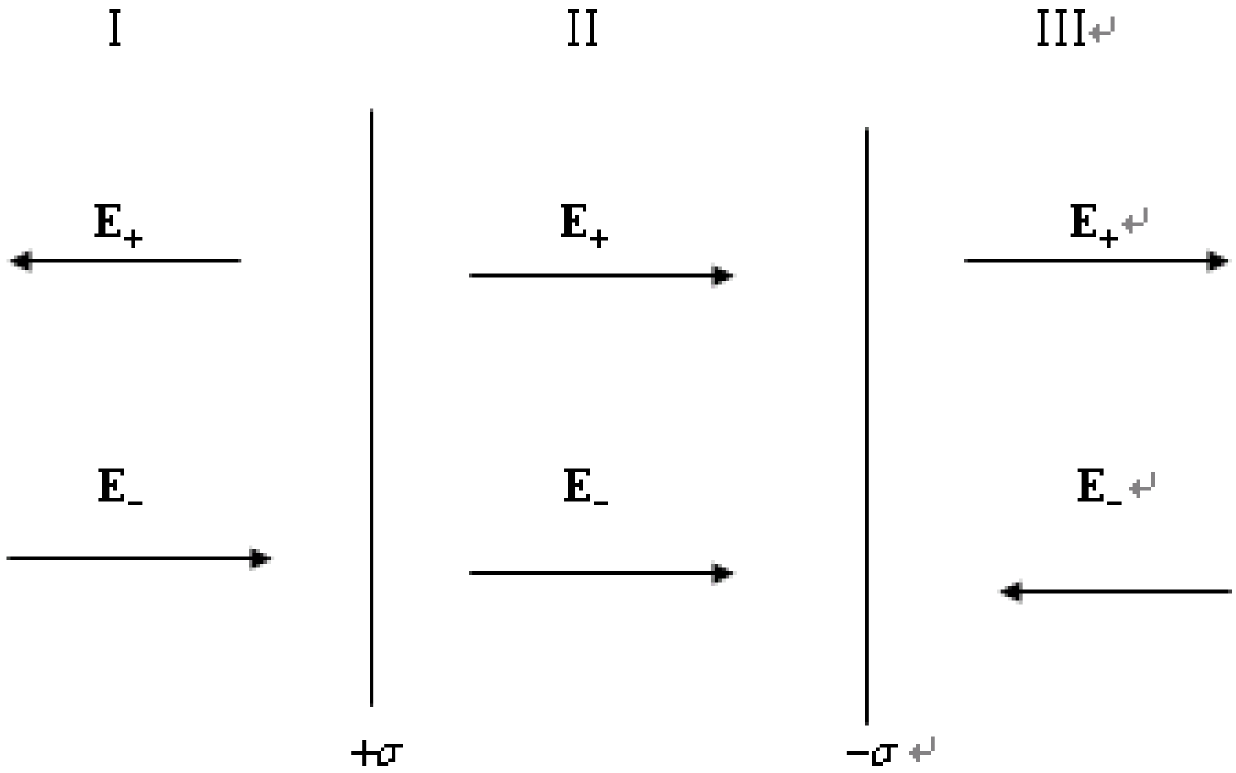
Or more compactly,

$$\boxed{\mathbf{E} = \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{n}}}$$

where $\hat{\mathbf{n}}$ is a unit vector pointing away from the plane.



•Two infinite parallel planes carrying equal but opposite uniform charge densities $\pm\sigma$



$\mathbf{E}_+$ and $\mathbf{E}_-$ are the E fields due to the positively and negatively charged plane, respectively.

They have equal magnitudes but opposite directions.

Therefore, E fields in region I and III vanish, while E field in region II

$$\mathbf{E} = 2 \times \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{n}} = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is a unit vector pointing from the positively charged plane to the negatively charged one.

Electrostatics

Electric Potential

The curl of E

Consider a point charge *q* at the origin.

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

Curl E is given by

$$\nabla\times\mathbf{E} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta E_\phi\right) - \frac{\partial E_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial E_r}{\partial\phi} - \frac{\partial}{\partial r}\left(rE_\phi\right)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rE_\theta\right) - \frac{\partial E_r}{\partial\theta}\right]\hat{\boldsymbol{\varphi}}$$

Here we have

$$E_r = \frac{q}{4\pi\varepsilon_0 r^2},\; E_\theta = E_\phi = 0$$

So,

$$\nabla \times \mathbf{E} = \frac{1}{r\sin\theta}\frac{\partial E_r}{\partial \phi}\hat{\boldsymbol{\theta}} - \frac{1}{r}\frac{\partial E_r}{\partial \theta}\hat{\boldsymbol{\varphi}} = \mathbf{0}$$

(For the origin, consider the line integral of a circle with radius r and q at the center. The line integral is obviously zero, and so is Curl E)

Electrostatic fields are Curl–free!

E field is conservative in electrostatics

For an arbitrary closed loop *C* enclosing an area *S* and not passing through the origin, from Stokes' theorem:

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = \int_S \left(\nabla\times\mathbf{E}\right)\cdot d\mathbf{a} = 0$$

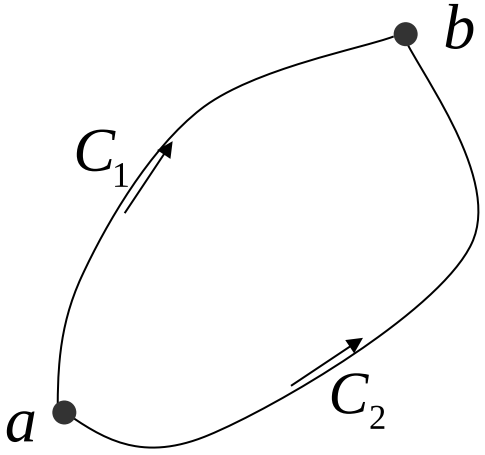
By the principle of superposition, for arbitrary charge distribution

$$\boxed{\oint_C \mathbf{E}\cdot d\mathbf{l} = 0}$$

This is equivalent to saying that the line integral from a point a to another point b is path independent.

Proof:

For two arbitrary paths C_1 *and* C_2 *joining* a *and* b



$$\oint \mathbf{E} \cdot d\mathbf{l} = \int_{C_1} \mathbf{E} \cdot d\mathbf{l} - \int_{C_2} \mathbf{E} \cdot d\mathbf{l} = 0$$

$$\therefore \int_{C_1} \mathbf{E} \cdot d\mathbf{l} = \int_{C_2} \mathbf{E} \cdot d\mathbf{l}$$

Electric Potential

Choose a reference point O and evaluate

$$\int_{\mathbf{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}$$

Since the integral is path independent, it is a function of r only. We can then define the electric potential by

$$\boxed{V(\mathbf{r}) = -\int_{\mathbf{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}}$$

Physical meaning:

$$V(\mathbf{r}) = -\int_{O}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l} = \int_{O}^{\mathbf{r}} \left(\frac{-\mathbf{F}}{1\mathrm{C}} \right) \cdot d\mathbf{l}$$

It is the work done by the external force in bringing one unit of charge from the reference point to r.

The potential difference between two points a and b is

$$V(\mathbf{b}) - V(\mathbf{a}) = \left(-\int_{\mathbf{O}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}\right) - \left(-\int_{\mathbf{O}}^{\mathbf{a}} \mathbf{E} \cdot d\mathbf{l}\right)$$

$$= -\left(\int_{\mathbf{O}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} - \int_{\mathbf{O}}^{\mathbf{a}} \mathbf{E} \cdot d\mathbf{l}\right)$$

$$= -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}$$

When b → a, we have

$$\Delta V = -\mathbf{E} \cdot d\mathbf{l}$$

Hence,

$$\boxed{\mathbf{E} = -\nabla V}$$

- **Unit:**

The unit of electric potential is

$$\frac{\text{Force}}{\text{Coulomb}} \times \text{Length} \quad = \mathbf{Nm/C} = \mathbf{J/C} = \mathbf{Volt}$$

Example:

What is the electric potential of a point charge q located at the origin, taking infinity as the reference point?

Answer:

Since the line integral of the E field is path independent, choose a radial path from infinity to the point r, which is at a distance *r* from the charge,

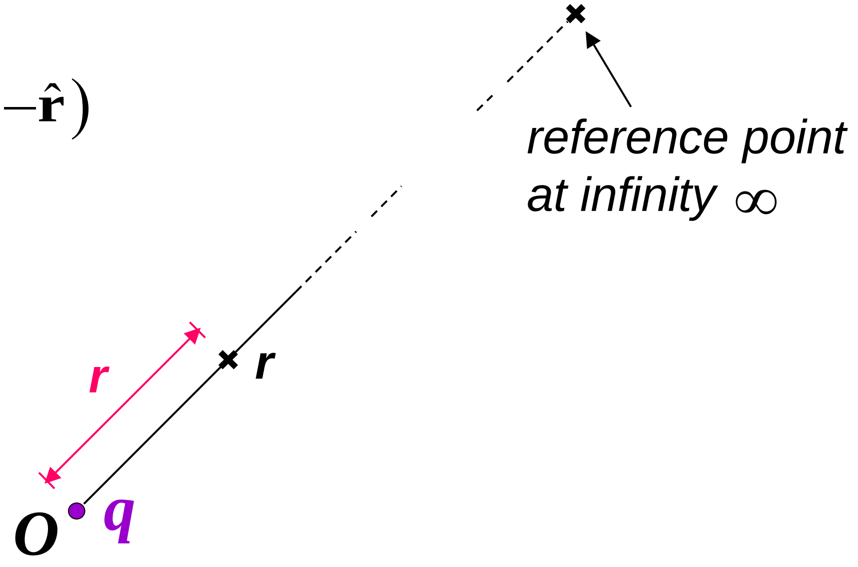
$$V(\mathbf{r}) = -\int_{\infty}^{\mathbf{r}} \mathbf{E}\cdot d\mathbf{l}$$

$$= -\int_{r}^{\infty} \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}\cdot dr\left(-\hat{\mathbf{r}}\right)$$

$$= \int_{r}^{\infty} \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\,dr$$

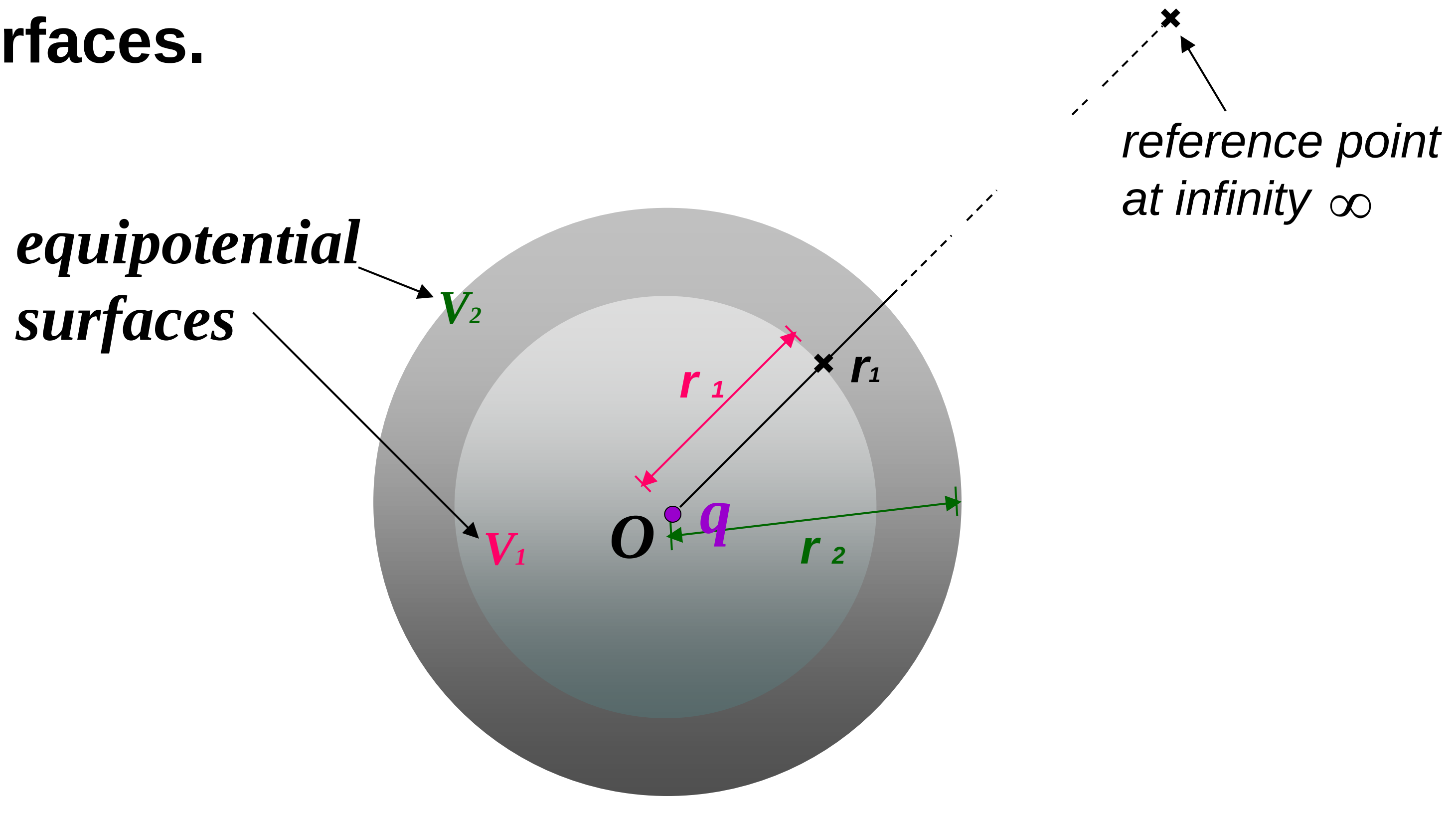
$$= \frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{r}\right]_r^{\infty}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$



Equipotential

A surface over which the potential is constant.
In the above example, the equipotential surfaces are surfaces with the same *r*, i.e., concentric spherical surfaces.



●Choice of reference point

In general, for a point charge at r', the potential at r is given by

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{4\pi\varepsilon_0}\frac{q}{\mathscr{r}}$$

In principle, the reference point can be chosen arbitrarily. Different choices of O yield *V* which differ only by a constant. The potential itself carries no real physical significance. Only potential difference matters.

However, notice that if the charge distribution extends to infinity, the choice of infinity as the reference point is not practical, as then the potential at any finite point will be infinite. Some other reference points should be chosen instead

Example:

What is the electric potential of an infinite line charge with linear charge density λ ?

Answer:

Use cylindrical coordinate and, without loss of generality, assume the line charge lies on the *z* axis. Since

$$\mathbf{E} = \frac{\lambda}{2\pi\varepsilon_0 s}\hat{\mathbf{s}}$$

if we had taken infinity as the reference point,

$$V(\mathbf{r}) = -\int_{\mathbf{r}}^{\infty} \frac{\lambda}{2\pi\varepsilon_0 s'}\hat{\mathbf{s}} \cdot ds'(-\hat{\mathbf{s}})$$

$$= \int_{s}^{\infty} \frac{\lambda}{2\pi\varepsilon_0 s'} ds'$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\left[\ln s'\right]_s^{\infty}$$

$$= \infty$$

Instead, take a point at a distance s_0 as the reference point.

Then

$$V(\mathbf{r}) = \int_{s}^{s_0} \frac{\lambda}{2\pi\varepsilon_0 s'} ds'$$

$$= \frac{\lambda}{2\pi\varepsilon_0} \left[\ln s'\right]_{s}^{s_0}$$

$$= \frac{\lambda}{2\pi\varepsilon_0} \ln \frac{s_0}{s}$$

Example:

What is the electric potential of an infinite surface charge with surface charge density σ ?

Answer:

Without loss of generality, assume the surface charge is on the *x*-*y* plane. Since

$$\mathbf{E} = \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{n}}$$

if we had taken infinity as the reference point, for a point r at a distance *d* above the *x*-*y* plane,

$$V(\mathbf{r}) = -\int_{\mathbf{r}}^{\infty} \frac{\sigma}{2\varepsilon_0} \hat{\mathbf{z}} \cdot dz' \left(-\hat{\mathbf{z}}\right)$$

$$= \int_{d}^{\infty} \frac{\sigma}{2\varepsilon_0} dz'$$

$$= \frac{\sigma}{2\varepsilon_0} \left[z'\right]_d^{\infty}$$

$$= \infty$$

Instead, take a point at a distance d_0 as the reference point.

Then

$$V(\mathbf{r}) = \int_{d}^{d_0} \frac{\sigma}{2\varepsilon_0} dz'$$

$$= \frac{\sigma}{2\varepsilon_0}\left[z'\right]_{d}^{d_0}$$

$$= \frac{\sigma}{2\varepsilon_0}\left(d_0 - d\right)$$

Superposition Principle and General Localized Charge Distributions

Since

$$V(\mathbf{r}) = -\int_{\mathbf{O}}^{\mathbf{r}} \left(\mathbf{E}_1 + \mathbf{E}_2 + \cdots\right) \cdot d\mathbf{l}$$

$$= -\int_{\mathbf{O}}^{\mathbf{r}} \mathbf{E}_1 \cdot d\mathbf{l} - \int_{\mathbf{O}}^{\mathbf{r}} \mathbf{E}_2 \cdot d\mathbf{l} - \cdots$$

$$= V_1(\mathbf{r}) + V_2(\mathbf{r}) + \cdots$$

In other words, *V* satisfies the principle of superposition.

Therefore, if infinity is taken as the reference point, for a collection of charges,

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \frac{q_i}{\mathscr{r}_i}$$

and for a *localized* continuous charge distribution

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{1}{\mathscr{r}} dq$$

Continuous charge distributions:

■**Charges along a line with linear charge density** $\lambda(\mathbf{r}')$

$$dq = \lambda dl' \rightarrow V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_P \frac{\lambda(\mathbf{r}')}{\mathscr{r}} dl'$$

■**Charges on a surface with surface charge density** $\sigma(\mathbf{r}')$

$$dq = \sigma da' \rightarrow V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\sigma(\mathbf{r}')}{\mathscr{r}} da'$$

■**Charges fill a volume with volume charge density** $\rho(\mathbf{r}')$

$$dq = \rho d\tau' \rightarrow V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{r}')}{\mathscr{r}} d\tau'$$

Example:

Calculate the potential inside and outside a uniformly charged sphere with charge q and radius R, by

(i) evaluating the work done in bringing one unit of charge of infinity to a point,

(ii) integrating the contributions due to the whole sphere.

Ans: (i) Recall that

$$\begin{cases} \mathbf{E}(r<R) = \dfrac{1}{4\pi\varepsilon_0}\dfrac{qr}{R^3}\hat{\mathbf{r}} \\ \mathbf{E}(r>R) = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{r^2}\hat{\mathbf{r}} \end{cases}$$

For *r* > *R*, the field is the same as if all the charges were located as a point charge at the center.

Therefore

$$V(r>R)=\frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

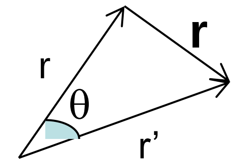
For *r* < *R*,

$$V(r<R)=-\int_{\infty}^{R}\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}dr-\int_{R}^{r}\frac{1}{4\pi\varepsilon_0}\frac{qr}{R^3}dr$$

$$=\frac{q}{4\pi\varepsilon_0 R}+\frac{q}{8\pi\varepsilon_0 R}\left(1-\frac{r^2}{R^2}\right)$$

$$=\frac{q}{8\pi\varepsilon_0 R}\left(3-\frac{r^2}{R^2}\right)$$

(ii)



$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}} \frac{\rho}{\mathscr{r}}\,d\tau$$

$$\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}'$$

$$\mathscr{r} = \sqrt{r^2 + r'^2 - 2rr'\cos\theta}$$

$$V(\mathbf{r}) = \frac{\rho}{4\pi\varepsilon_0}\int_{\mathcal{V}} \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}}\, r'^2 \sin\theta\, dr'\, d\theta\, d\phi$$

$$= \frac{\rho}{4\pi\varepsilon_0}\int_0^{2\pi} d\phi \int_0^R r'^2 dr' \int_0^{\pi} \frac{\sin\theta}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}}\, d\theta$$

$$= \frac{\rho}{2\varepsilon_0}\int_0^R r'^2 dr' \int_0^{\pi} \frac{\sin\theta}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}}\, d\theta$$

The θ integral yields

$$\int_0^{\pi} \frac{\sin\theta}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}}\, d\theta = \begin{cases} \dfrac{2}{r} & \text{for } r > r' \\ \dfrac{2}{r'} & \text{for } r' > r \end{cases}$$

(Refer to the supplementary notes for details)

Therefore, for observation points outside the sphere,

$$V(\mathbf{r}) = \frac{\rho}{2\varepsilon_0}\int_0^R r'^2 dr' \times \frac{2}{r}$$

$$= \frac{\rho}{\varepsilon_0 r}\frac{R^3}{3} = \frac{1}{\varepsilon_0 r}\rho \times \frac{4\pi R^3}{3} \times \frac{1}{4\pi}$$

$$= \frac{q}{4\pi\varepsilon_0 r}$$

For observation points inside the sphere

$$V(\mathbf{r}) = \frac{\rho}{2\varepsilon_0}\left[\int_0^r r'^2 dr' \times \frac{2}{r} + \int_r^R r'^2 dr' \times \frac{2}{r'}\right]$$

$$= \frac{\rho}{\varepsilon_0}\left[\frac{1}{r}\int_0^r r'^2 dr' + \int_r^R r' dr'\right]$$

$$= \frac{\rho}{\varepsilon_0}\left[\frac{1}{r}\frac{r^3}{3} + \frac{R^2 - r^2}{2}\right]$$

$$= \frac{\rho R^3}{2\varepsilon_0}\left(\frac{1}{R} - \frac{r^2}{3R^3}\right)$$

$$= \frac{q}{8\pi\varepsilon_0 R}\left(3 - \frac{r^2}{R^2}\right)$$

Poisson's and Laplace's Equation

Since

$$\boxed{\mathbf{E} = -\nabla V}$$

We have

$$\nabla \cdot \mathbf{E} = \nabla \cdot \left(-\nabla V \right) = -\nabla^2 V$$

From Gauss's law,

$$\boxed{\nabla^2 V = -\frac{\rho}{\varepsilon_0}}$$

This is the Poisson's equation.

In particular, in a charge-free region, $\rho = 0$ **,**

and we have

$$\boxed{\nabla^2 V = 0}$$

which is called the Laplace's equation.

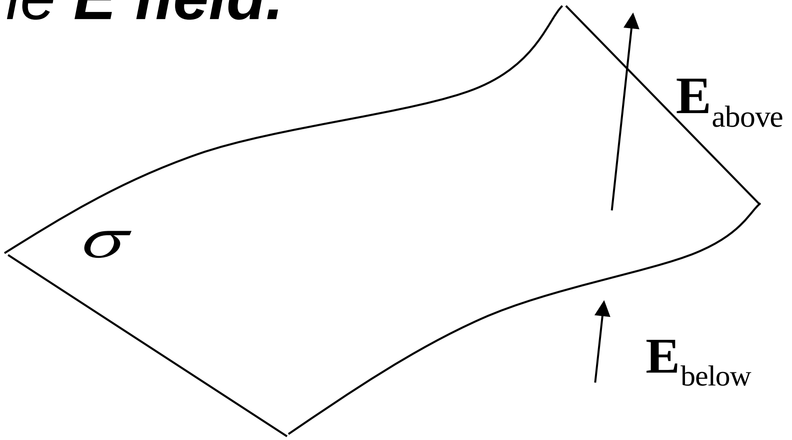
Electrostatics

Boundary Conditions

Boundary Condition Across Surface

Consider a surface with surface charge density σ.

Let the ***E field*** *below and above the surface be* $\mathbf{E}_{\text{below}}$ *and* $\mathbf{E}_{\text{above}}$*, respectively. The surface density may in general vary from point to point on the surface, and so does the* ***E field.***

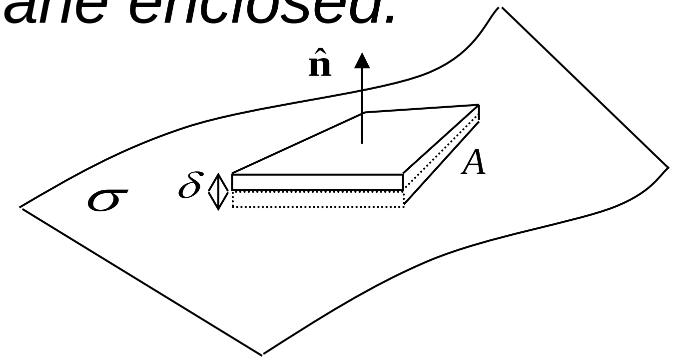


Boundary Condition Across Surface

- **Perpendicular component of E field**

Consider a "pillbox" Gaussian surface with top and bottom surface area A and height δ

The area A is assumed to be very small so that the field on the top and bottom surface of the pillbox is approximately constant, and so is the surface charge density of the plane enclosed.



Boundary Condition Across Surface

The flux through the lateral surfaces can be neglected when we take $\delta \to 0$*.*
Therefore, from Gauss's law, we have

$$\mathbf{E}_{\text{above}} \cdot A\hat{\mathbf{n}} + \mathbf{E}_{\text{below}} \cdot A(-\hat{\mathbf{n}}) = \frac{\sigma A}{\varepsilon_0}$$

$$\boxed{\left(\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}}\right) \cdot \hat{\mathbf{n}} = \frac{\sigma}{\varepsilon_0}}$$

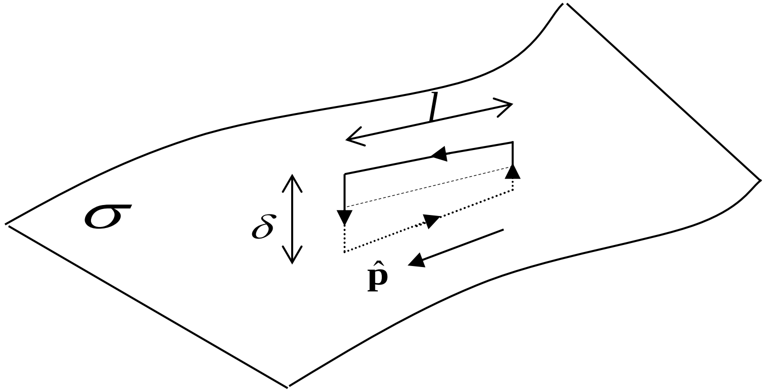
where $\hat{\mathbf{n}}$ *is a unit vector perpendicular to the plane and pointing from "below" to "above"*

Boundary Condition Across Surface

- **Parallel component of E field**

Consider a thin rectangular loop with length l *and width* δ *with one side above and the other below the surface.*

The long sides of the rectangle are along the direction of a unit vector $\hat{\mathbf{p}}$ *, which lies on the surface.*



Boundary Condition Across Surface

*Consider the line integral of the **E field** along the rectangular loop. If we take* $\delta \to 0$ *such that the contributions of the two short sides can be neglected.*

Assume that l *is very small such that the **E field** along the long sides is approximately constant, we have*

$$\mathbf{E}_{\text{above}} \cdot l\hat{\mathbf{p}} + \mathbf{E}_{\text{below}} \cdot l\left(-\hat{\mathbf{p}}\right) = 0$$

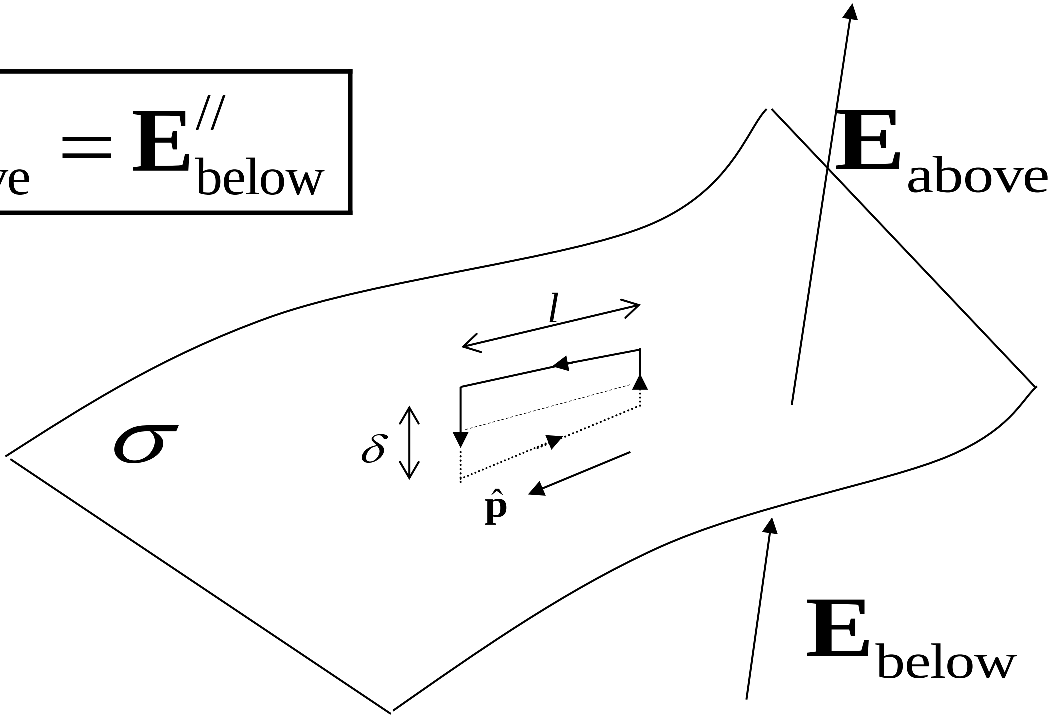
$$\mathbf{E}_{\text{above}} \cdot \hat{\mathbf{p}} = \mathbf{E}_{\text{below}} \cdot \hat{\mathbf{p}}$$

The above equation holds because the field is curl-free.

Boundary Condition Across Surface

Since this is true for any $\hat{\mathbf{p}}$ *on the surface, we have*

$$\mathbf{E}_{\text{above}}^{//} = \mathbf{E}_{\text{below}}^{//}$$



Boundary Condition Across Surface

E Field Across a Surface

In conclusion, we have

$$\boxed{\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{n}}}$$

Potential Across a Surface

Since

$$V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}$$

and the E field is finite everywhere, therefore the potential is continuous across the surface, i.e.,

$$\boxed{V_{\text{above}} = V_{\text{below}}}$$

However, because

$$\mathbf{E} = -\nabla V$$

we have

$$\left(\nabla V_{\text{above}} - \nabla V_{\text{below}}\right) \cdot \hat{\mathbf{n}} = -\frac{\sigma}{\varepsilon_0}$$

$$\left(\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}}\right) \cdot \hat{\mathbf{n}} = \frac{\sigma}{\varepsilon_0}$$

Potential Across a Surface

Denote the normal derivative of *V*, $\nabla V \cdot \hat{\mathbf{n}}$**, by** $\frac{\partial V}{\partial n}$ **,**

we have

$$\boxed{\frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n} = -\frac{\sigma}{\varepsilon_0}}$$

Electrostatics

Work and Energy

To move a charge Q from $\mathbf{a}$ to $\mathbf{b}$:

Electric force = $Q\mathbf{E}$

External force applied = $-Q\mathbf{E}$

Work done $$W = -Q\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E}\cdot d\mathbf{l} = Q\left[V\left(\mathbf{b}\right) - V\left(\mathbf{a}\right)\right]$$

$$\therefore V\left(\mathbf{b}\right) - V\left(\mathbf{a}\right) = \frac{W}{Q}$$

The potential difference between $\mathbf{a}$ and $\mathbf{b}$ is the work done to move one unit of charge from $\mathbf{a}$ to $\mathbf{b}$.

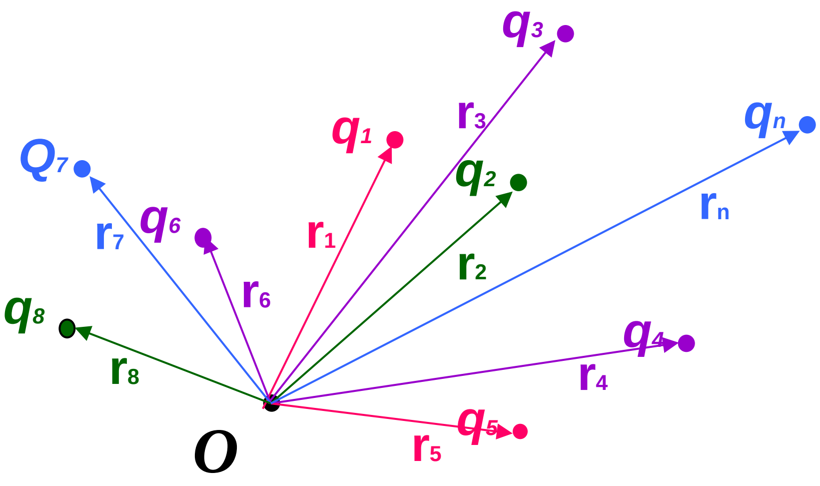
In particular, if a is the reference point,

$$V(\mathbf{b}) - V(\mathbf{O}) = V(\mathbf{b}) = \frac{W}{Q}$$

The potential of point b is the work done to move one unit of charge from the reference point to b.

If $\mathbf{O} = \infty$ ***,potential is the work it takes to create the system (potential energy) per unit charge***

Suppose we have a number of discrete point charges $q_1, q_2, \cdots, q_n$ **located at** $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_n$ **, respectively.**



Consider the energy required to create this system

1.Move q_1 from ∞ to $\mathbf{r}_1$, no work done required.

2.Move q_2 from ∞ to $\mathbf{r}_2$, work done =

$$q_2 \times \frac{1}{4\pi\varepsilon_0}\frac{q_1}{\mathscr{r}_{12}} \qquad \textbf{where } \mathscr{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1$$

3.Move q_3 from ∞ to $\mathbf{r}_3$, work done =

$$q_3 \times \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{\mathscr{r}_{13}} + \frac{q_2}{\mathscr{r}_{23}}\right) \qquad \textbf{where } \mathscr{r}_{13} = \mathbf{r}_3 - \mathbf{r}_1 \textbf{ , } \mathscr{r}_{23} = \mathbf{r}_3 - \mathbf{r}_2$$

In general, in moving q_i from ∞ to $\mathbf{r}_i$, work done

$$W_i = q_i \times \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r_{1i}} + \frac{q_2}{r_{2i}} + \cdots + \frac{q_{i-1}}{r_{i-1,i}}\right) = q_i \times \frac{1}{4\pi\varepsilon_0}\sum_{j=1}^{i-1}\frac{q_j}{r_{ji}}$$

, where $r_{ji} = \mathbf{r}_i - \mathbf{r}_j$

Therefore, total work done in creating the system

$$W = \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} q_i \times \frac{1}{4\pi\varepsilon_0}\sum_{j=1}^{i-1}\frac{q_j}{r_{ji}} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{j=1}^{i-1}\frac{q_i q_j}{r_{ji}}$$

Notice that $r_{ij} = r_{ji}$, therefore we can "double count" the terms and then divide the work done by 2:

$$W = \frac{1}{8\pi\varepsilon_0} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{q_i q_j}{r_{ij}}$$

Notice that we don't take into account the energy required to create the point charges. We assume that they are given to us ready-made.

One can rewrite the above expression as

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{4\pi\varepsilon_0}\frac{q_j}{r_{ij}} = \frac{1}{2}\sum_{i=1}^{n} q_i V\left(\mathbf{r}_i\right)$$

For continuous charge distribution, discretize the distribution, consider them as a set of discrete point charges and use the above equation, we have

$$W = \frac{1}{2}\int_{\mathcal{V}} \rho V d\tau$$

Note:

- During the discretization, we also ignore the work done to create each small volume element, and only consider the interaction between different volume elements.

  However, as we further discretize the volume element, the ignored energy will be taken into account.

  Therefore we obtain the “real” total energy in the expression.

Note:

- **The volume of integral $\mathcal{V}$ is arbitrary as long as it includes all the charges. Because $\rho = 0$ in charge-free regions and does not contribute to the integral. In particular, we can take $\mathcal{V}$ as the whole space.**

$$W = \frac{1}{2}\int_{\mathcal{V}} \rho V d\tau$$

From Gauss's law

$$\rho = \varepsilon_0 \nabla \cdot \mathbf{E}$$

Therefore,

$$W = \frac{1}{2}\varepsilon_0 \int_{\mathcal{V}} (\nabla \cdot \mathbf{E}) V d\tau$$

Recall that

$$\nabla \cdot (V\mathbf{E}) = V(\nabla \cdot \mathbf{E}) + \mathbf{E} \cdot \nabla V$$

$$= V(\nabla \cdot \mathbf{E}) - E^2$$

We have

$$W = \frac{1}{2}\varepsilon_0 \int_{\mathcal{V}} \left[\nabla \cdot (V\mathbf{E}) + E^2 \right] d\tau$$

From divergence theorem

$$\int_{\mathcal{V}} \nabla \cdot (V\mathbf{E})\, d\tau = \oint_S V\mathbf{E} \cdot d\mathbf{a}$$

where S is the closed surface enclosing $\mathcal{V}$

If we take $\mathcal{V}$ as the entire space, the integral vanishes because

$$V \sim \frac{1}{r} \quad \Rightarrow \quad E \sim \frac{1}{r^2}$$

and the area of the surface $A \propto r^2$

Hence,

$$\boxed{W = \frac{1}{2}\varepsilon_0 \int_{\text{all space}} E^2 d\tau}$$

We therefore interpret $\frac{1}{2}\varepsilon_0 E^2$ ***as the energy density of the electrostatic field.***

Example:

Calculate the energy required to create a uniformly charged sphere with radius R and charge q by using the relations (i)

$$W = \frac{1}{2}\int_{\nu} \rho V d\tau$$

and (ii)

$$W = \frac{1}{2}\varepsilon_0 \int_{\text{all space}} E^2 d\tau$$

$$W = \frac{1}{2}\int_{\nu} \rho V d\tau$$

Ans:

(i) Recall that

$$\begin{cases} V(r > R) = \dfrac{q}{4\pi\varepsilon_0 r} \\ V(r < R) = \dfrac{q}{8\pi\varepsilon_0 R}\left(3 - \dfrac{r^2}{R^2}\right) \end{cases}$$

$$W = \frac{1}{2}\int_{\mathcal{V}} \rho V d\tau$$

Therefore,

$$W = \frac{1}{2}\int_{\mathcal{V}} \rho \frac{q}{8\pi\varepsilon_0 R}\left(3 - \frac{r^2}{R^2}\right) d\tau$$

$$= \frac{\rho q}{16\pi\varepsilon_0 R}\left[3 \times \frac{4\pi R^3}{3} - \frac{1}{R^2}\int_0^R r^2 \times 4\pi r^2 dr\right]$$

$$= \frac{\rho q}{16\pi\varepsilon_0 R}\left[3 \times \frac{4\pi R^3}{3} - \frac{4\pi}{R^2}\frac{R^5}{5}\right]$$

$$= \frac{q}{16\pi\varepsilon_0 R}\rho \times \frac{4\pi R^3}{3}\left[3 - \frac{3}{5}\right]$$

$$= \frac{q^2}{16\pi\varepsilon_0 R} \times \frac{12}{5}$$

$$= \frac{3q^2}{20\pi\varepsilon_0 R}$$

$$W = \frac{1}{2}\varepsilon_0 \int_{\text{all space}} E^2 d\tau$$

(ii) Recall that

$$\begin{cases} \mathbf{E}(r < R) = \dfrac{1}{4\pi\varepsilon_0} \dfrac{qr}{R^3} \hat{\mathbf{r}} \\ \mathbf{E}(r > R) = \dfrac{1}{4\pi\varepsilon_0} \dfrac{q}{r^2} \hat{\mathbf{r}} \end{cases}$$

$$W = \frac{1}{2}\varepsilon_0 \int_{\text{all space}} E^2 d\tau$$

So,

$$W_{\text{in}} = \frac{\varepsilon_0}{2}\int_0^R \left(\frac{q}{4\pi\varepsilon_0 R^3}\right)^2 r^2\, 4\pi r^2 dr$$

$$= \frac{q^2}{8\pi\varepsilon_0 R^6}\int_0^R r^4 dr$$

$$= \frac{q^2}{40\pi\varepsilon_0 R}$$

$$W_{\text{out}} = \frac{\varepsilon_0}{2}\int_R^\infty \left(\frac{q}{4\pi\varepsilon_0}\right)^2 \frac{1}{r^4}\, 4\pi r^2 dr$$

$$= \frac{q^2}{8\pi\varepsilon_0}\int_R^\infty \frac{1}{r^2}\, dr$$

$$= \frac{q^2}{8\pi\varepsilon_0 R}$$

$$W = W_{\text{in}} + W_{\text{out}} = \frac{3q^2}{20\pi\varepsilon_0 R}$$

- ***For a point charge q, R → 0, and*** $W = \infty$ ***!!!***

Electrostatics

Conductors

Basic Properties of Conductors

I. *A conductor contains an unlimited supply of completely free charges. If* $\mathbf{E} \neq \mathbf{0}$ → *current* → *No electrostatics.*

Therefore, ***E = 0 inside a conductor***

II. ***E = 0 inside a conductor*** → *For any two points* ***a and b within or on the surface,***

$$V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} = 0$$

Therefore, The whole conductor, including the surface, is an equipotential.

Basic Properties of Conductors

III. *If the **E field** on the surface has non-zero tangential component → Current flow on the surface → No electrostatics. Therefore,*
E field on the surface is perpendicular to the surface

IV. *From Gauss's law,*

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$$

$$\mathbf{E} = \mathbf{0} \Rightarrow \rho = 0$$

$\rho = 0$ ***inside a conductor***

V. $\rho = 0$ *inside a conductor. Therefore,*
Any net charge resides on the surface

Basic Properties of Conductors

VI. *The **E field** just outside the conductor is*

$$\mathbf{E} = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{n}}$$

, where

σ *is the surface charge density of the conductor and*
$\hat{\mathbf{n}}$ *is a unit vector normal to the surface and pointing* ***"outward"*** *from the conductor.*

Proof of (III):

(E field on the surface is perpendicular to the surface)

Since the E field inside a conductor = 0 and the parallel component of E field is continuous across the surface,

therefore, the E field just outside the surface has perpendicular component only.

Proof of (VI)

($\rho = 0$ inside a conductor)

Since the E field inside a conductor = 0 and on the surface:

$$\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{n}}$$

therefore the E field just outside the conductor is

$$\mathbf{E} = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{n}}$$

Potential Across the Surface

Recall that for an arbitrary surface ,

we have

$$\boxed{\frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n} = -\frac{\sigma}{\varepsilon_0}}$$

- ***For a conductor, since V is constant inside, and*** $\mathbf{E} = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{n}}$***, we have***

$$\boxed{\frac{\partial V}{\partial n} = -\frac{\sigma}{\varepsilon_0}}$$

Capacitors

Consider two conductors with charge +Q on one and –Q on the other.

Since the potentials are constant in each conductor, one can define the potential difference between the two conductors.

Capacitors

since

$$V_+ - V_- = -\int_-^+ \mathbf{E} \cdot d\mathbf{l}$$

and

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \int \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^2} \right) \rho \, d\tau$$

We have

$$V \propto E \propto \rho \propto Q$$

Define the **capacitance** by

$$C = \frac{Q}{V}$$

- ***By definition, Q is positive, V is the potential difference of the positively charged conductor relative to the negatively charged one, and therefore is also positive.***

  So, C is by definition positive.

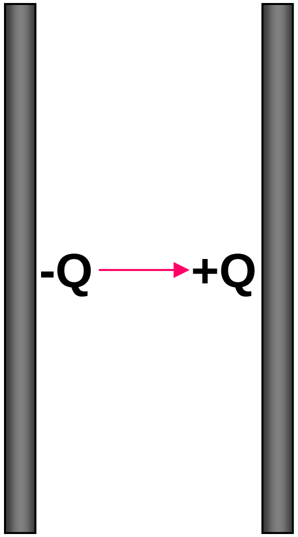
- ***The capacitance of a single conductor is similarly defined, by assuming an imaginary surrounding spherical shell of infinite radius as the second conductor.***

- ***Capacitance is a pure geometrical quantity***

- ***The (mks) unit of capacitance is farads (F), defined by F = C/V***

Energy Stored in a Capacitor

Consider two neutral conductors. One has to move an amount of charge Q from the negative conductor to the positive one.



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During the process, when the charge moved is q, since C is independent of the q, the potential difference is V = q/C,

and the work done in moving another dq is dW = V dq, therefore

$$W = \int_0^Q \frac{q}{C} dq = \frac{1}{2}\frac{Q^2}{C}$$

or

$$\boxed{W = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2}$$

Example:

(i) Find the capacitance of a parallel-plate capacitor consisting of two metal surfaces of area A and held a distance d apart.

(ii) Find the energy stored in the capacitor by using

$$W = \frac{1}{2}\varepsilon_0 \int_{\text{all space}} E^2 d\tau$$

Ans:

(i) Assuming that the plates can be considered infinite, the magnitude of the E field in between the plate given by

$$E = \frac{\sigma}{\varepsilon_0}$$

Therefore,

$$V = Ed = \frac{\sigma d}{\varepsilon_0} = Q\frac{d}{\varepsilon_0 A}$$

Hence,

$$\boxed{C = \frac{\varepsilon_0 A}{d}}$$

Ans:

(ii)Since the field is constant in between the plates

$$W = \frac{\varepsilon_0}{2} E^2 Ad = \frac{\varepsilon_0}{2}\left(\frac{\sigma}{\varepsilon_0}\right)^2 Ad = \frac{1}{2}\frac{d}{\varepsilon_0 A}(\sigma A)^2$$

$$= \frac{1}{2}\frac{Q^2}{C}$$