VECTOR CALCULUS

Differentiation of vectors

Consider a vector $\mathbf{a}(u)$ that is a function of a scalar variable u.

The derivative of $\mathbf{a}(u)$ with respect to u is defined as

$$\frac{d\mathbf{a}}{du} = \lim_{\Delta u \to 0} \frac{\mathbf{a}(u + \Delta u) - \mathbf{a}(u)}{\Delta u}. \qquad (1)$$

Note that $\frac{d\mathbf{a}}{du}$ is also a vector, which is not, in general, parallel to $\mathbf{a}(u)$.

Example

The position vector of a particle at time t in Cartesian coordinates is given by $\mathbf{r}(t) = 2t^2\mathbf{i} + (3t-2)\mathbf{j} + (3t^2-1)\mathbf{k}$. Find the speed of the particle at $t = 1$, and the component of its acceleration in the direction $\mathbf{s} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

Answer

The velocity and acceleration of the particle are given by

$$\begin{aligned} \mathbf{v}(t) = \frac{d\mathbf{r}}{dt} &= 4t\mathbf{i} + 3\mathbf{j} + 6t\mathbf{k} \\ \mathbf{a}(t) = \frac{d\mathbf{v}}{dt} &= 4\mathbf{i} + 6\mathbf{k} \end{aligned}$$

The speed of the particle at $t = 1$ is

$$|\mathbf{v}(1)| = \sqrt{4^2 + 3^2 + 6^2} = \sqrt{61}$$

The acceleration of the particle is constant (independent of t), and its component in the direction $\mathbf{s}$ is

$$\mathbf{a} \cdot \hat{\mathbf{s}} = \frac{(4\mathbf{i} + 6\mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} + \mathbf{k})}{\sqrt{1^2 + 2^2 + 1^2}} = \frac{5\sqrt{6}}{3}$$

Example

The position vector of a particle in plane polar coordinates is $\mathbf{r}(t) = \rho(t)\hat{\mathbf{e}}_\rho$. Find expressions for the velocity and acceleration of the particle in these coordinates.

Answer

The velocity is given by

$$\mathbf{v}(t) = \dot{\mathbf{r}}(t) = \dot{\rho}\hat{\mathbf{e}}_\rho + \rho\dot{\hat{\mathbf{e}}}_\rho = \dot{\rho}\hat{\mathbf{e}}_\rho + \rho\dot{\phi}\hat{\mathbf{e}}_\phi$$

since

$$\frac{d\hat{\mathbf{e}}_\rho}{dt} = -\sin\phi\frac{d\phi}{dt}\mathbf{i} + \cos\phi\frac{d\phi}{dt}\mathbf{j} = \dot{\phi}\hat{\mathbf{e}}_\phi$$

The acceleration is given by

$$\begin{aligned}
\mathbf{a}(t) &= \frac{d}{dt}(\dot{\rho}\hat{\mathbf{e}}_\rho + \rho\dot{\phi}\hat{\mathbf{e}}_\phi) \\
&= \ddot{\rho}\hat{\mathbf{e}}_\rho + \dot{\rho}\dot{\hat{\mathbf{e}}}_\rho + \rho\dot{\phi}\dot{\hat{\mathbf{e}}}_\phi + \rho\ddot{\phi}\hat{\mathbf{e}}_\phi + \dot{\rho}\dot{\phi}\hat{\mathbf{e}}_\phi \\
&= \ddot{\rho}\hat{\mathbf{e}}_\rho + \dot{\rho}(\dot{\phi}\hat{\mathbf{e}}_\phi) + \rho\dot{\phi}(-\dot{\phi}\hat{\mathbf{e}}_\rho) + \rho\ddot{\phi}\hat{\mathbf{e}}_\phi + \dot{\rho}\dot{\phi}\hat{\mathbf{e}}_\phi \\
&= (\ddot{\rho} - \rho\dot{\phi}^2)\hat{\mathbf{e}}_\rho + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})\hat{\mathbf{e}}_\phi
\end{aligned}$$

since

$$\frac{d\hat{\mathbf{e}}_\phi}{dt} = -\cos\phi\frac{d\phi}{dt}\mathbf{i} - \sin\phi\frac{d\phi}{dt}\mathbf{j} = -\dot{\phi}\hat{\mathbf{e}}_\rho$$

Differentiation of composite vector expressions

Assuming $\mathbf{a}$ and $\mathbf{b}$ are differentiable vector functions of a scalar u, and ϕ is a differentiable scalar function of u:

$$\frac{d}{du}(\phi\mathbf{a}) = \phi\frac{d\mathbf{a}}{du} + \frac{d\phi}{du}\mathbf{a} \quad (2)$$

$$\frac{d}{du}(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot\frac{d\mathbf{b}}{du} + \frac{d\mathbf{a}}{du}\cdot\mathbf{b} \quad (3)$$

$$\frac{d}{du}(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times\frac{d\mathbf{b}}{du} + \frac{d\mathbf{a}}{du}\times\mathbf{b} \quad (4)$$

If a vector $\mathbf{a}(s)$ is a function of the scalar variable s, which is itself a function of u such that $s = s(u)$, then we have

$$\frac{d\mathbf{a}(s)}{du} = \frac{ds}{du}\frac{d\mathbf{a}}{ds}$$

Integration of vectors

Example

A small particle of mass m orbits a much larger mass M centered at the origin O. According to Newton's law of gravitation, the position vector $\mathbf{r}$ of the small mass obeys the differential equation

$$m\frac{d^2\mathbf{r}}{dt^2} = -\frac{GMm}{r^2}\hat{\mathbf{r}}$$

Show that the vector $\mathbf{r} \times d\mathbf{r}/dt$ is a constant of motion.

Answer

Forming the vector product of the differential equation with $\mathbf{r}$, we obtain

$$\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} = -\frac{GM}{r^2}\mathbf{r} \times \hat{\mathbf{r}} = \mathbf{0}$$

But

$$\frac{d}{dt}\left(\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right)=\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}+\frac{d\mathbf{r}}{dt}\times\frac{d\mathbf{r}}{dt}=\mathbf{0}$$

Integrating,

$$\mathbf{r}\times\frac{d\mathbf{r}}{dt}=\mathbf{c} \tag{5}$$

where $\mathbf{c}$ is a constant vector.

In an infinitesimal time dt the change in position vector of the small mass is $d\mathbf{r}$ and the element of area swept out by the position vector of the particle is $dA=\frac{1}{2}|\mathbf{r}\times d\mathbf{r}|$. Dividing by dt, we obtain

$$\frac{dA}{dt}=\frac{1}{2}\left|\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right|=\frac{|\mathbf{c}|}{2}$$

Therefore, the physical interpretation of Eq. 5 is that the position vector $\mathbf{r}$ of the small mass sweeps out equal areas in equal times.

Space curves

A curve C can be described by the vector $\mathbf{r}(u)$ joining the origin O of a coordinate system to a point on the curve.

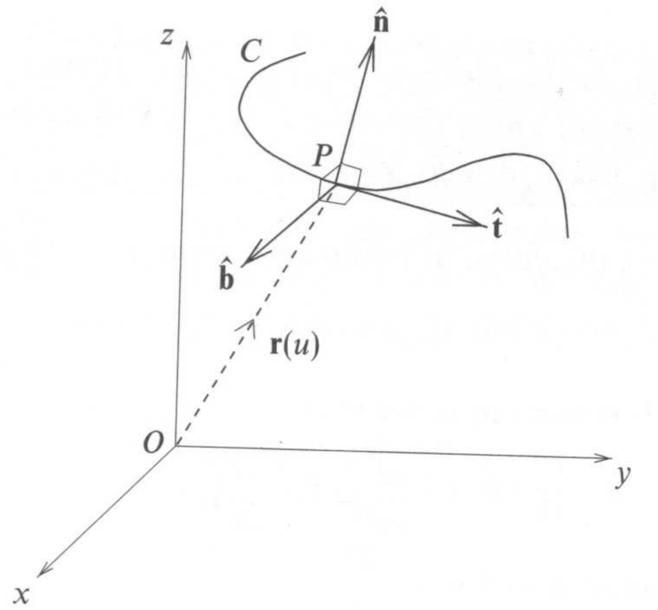

FIG. 1: The unit tangent $\hat{\mathbf{t}}$, normal $\hat{\mathbf{n}}$ and binormal $\hat{\mathbf{b}}$ to the space curve C at a particular point P.

As the parameter u varies, the end-point of the vector moves along the curve. In Cartesian coordinates,

$$\mathbf{r}(u) = x(u)\mathbf{i} + y(u)\mathbf{j} + z(u)\mathbf{k}$$

where $x = x(u)$, $y = y(u)$, $z = xz(u)$ are the parametric equations of the curve.

A curve may be described in parametric form by the vector $\mathbf{r}(s)$, where the parameter s is the arc length along the curve measured from a fixed point. For the curve described by $\mathbf{r}(u)$, consider the infinitesimal vector displacement

$$d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$$

along the curve. The square of this distance moved is

$$(ds)^2 = d\mathbf{r} \cdot d\mathbf{r} = (dx)^2 + (dy)^2 + (dz)^2$$

so that

$$\left(\frac{ds}{du}\right)^2 = \frac{d\mathbf{r}}{du} \cdot \frac{d\mathbf{r}}{du}$$

Therefore, the arc length between two points on the curve $\mathbf{r}(u)$, given by $u = u_1$ and $u = u_2$, is

$$s = \int_{u_1}^{u_2} \sqrt{\frac{d\mathbf{r}}{du} \cdot \frac{d\mathbf{r}}{du}} du \qquad (6)$$

If a curve C is described by $\mathbf{r}(u)$, then $d\mathbf{r}/ds$ is a unit tangent vector to C and its denoted by $\hat{\mathbf{t}}$.

The rate at which $\hat{\mathbf{t}}$ changed with respect to s is given by $d\hat{\mathbf{t}}/ds$, and its magnitude is defined as the curvature κ of the curve C at a given point,

$$\kappa = \left|\frac{d\hat{\mathbf{t}}}{ds}\right| = \left|\frac{d^2\hat{\mathbf{r}}}{ds^2}\right|.$$

We can also define the quantity $\rho = 1/\kappa$, which is called the radius of curvature. Note that $d\hat{\mathbf{t}}/ds$ is perpendicular to $\hat{\mathbf{t}}$, and its unit vector direction is denoted by $\hat{\mathbf{n}}$ (principal normal). We therefore have

$$\frac{d\hat{\mathbf{t}}}{ds} = \kappa\hat{\mathbf{n}} \qquad (7)$$

The unit vector $\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}}$, which is perpendicular to the plane containing $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$, is called the binormal to C. The rate at which $\hat{\mathbf{b}}$ changes with respect to s is given by $d\hat{\mathbf{b}}/ds$. In particular,

$$\frac{d\hat{\mathbf{b}}}{ds} = -\tau\hat{\mathbf{n}}. \qquad (8)$$

Vector functions of several arguments

If $\mathbf{a} = \mathbf{a}(u_1, u_2, \ldots, u_n)$ and each of the u_i is also a function $u_i(v_1, v_2, \ldots, v_n)$ of the variables v_i, then

$$\begin{aligned} \frac{\partial \mathbf{a}}{\partial v_i} &= \frac{\partial \mathbf{a}}{\partial u_1}\frac{\partial u_1}{\partial v_i} + \frac{\partial \mathbf{a}}{\partial u_2}\frac{\partial u_2}{\partial v_i} + \cdots + \frac{\partial \mathbf{a}}{\partial u_n}\frac{\partial u_n}{\partial v_i} \\ &= \sum_{j=1}^{n} \frac{\partial \mathbf{a}}{\partial u_j}\frac{\partial u_j}{\partial v_i} \qquad (9) \end{aligned}$$

A special case of this rule arises when $\mathbf{a}$ is an explicit function of some variable v, as well as of scalars $u_1, u_2, \ldots, u_n$ that are themselves functions of v. Then we have

$$\frac{d\mathbf{a}}{dv} = \frac{\partial \mathbf{a}}{\partial v} + \sum_{j=1}^{n} \frac{\partial \mathbf{a}}{\partial u_j}\frac{\partial u_j}{\partial v} \qquad (10)$$

Surfaces

A surface S can be described by the vector $\mathbf{r}(u, v)$ joining the origin O of a coordinate system to a point on the curve.

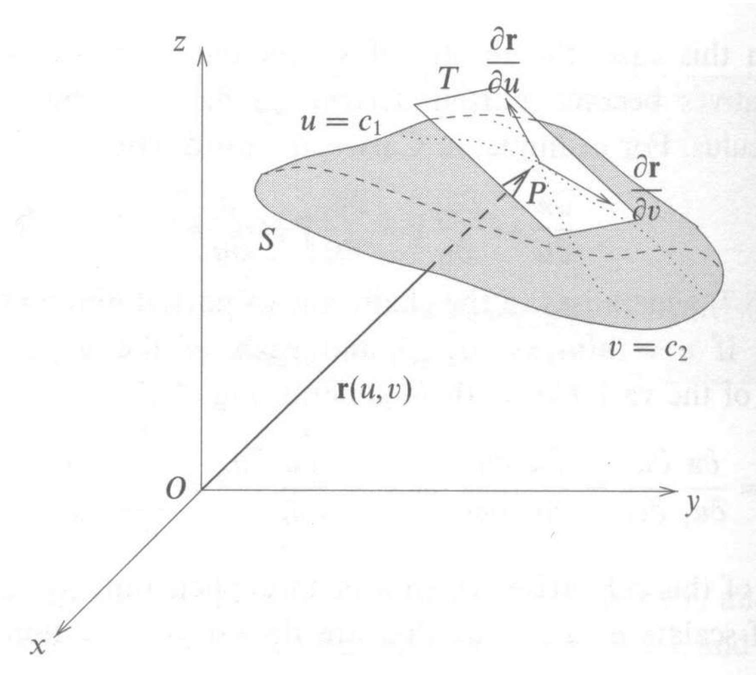

FIG. 2: The tangent plane T to a surface S at a particular point P; $u = c_1$ and $v = c_2$ are the coordinate curves.

As the parameters u and v vary, the end-point of the vector moves over the surface.

In Cartesian coordinates, the surface is given by

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

If the surface is smooth, then at any point P on S the vectors $\partial\mathbf{r}/\partial u$ and $\partial\mathbf{r}/\partial v$ are linearly independent, and define the tangent plane T at the point P. A vector normal to the surface at P is given by

$$\mathbf{n} = \frac{\partial\mathbf{r}}{\partial u} \times \frac{\partial\mathbf{r}}{\partial v}$$

In the neighbourhood of P, an infinitesimal vector displacement $d\mathbf{r}$ is written as

$$d\mathbf{r} = \frac{\partial\mathbf{r}}{\partial u}du + \frac{\partial\mathbf{r}}{\partial v}dv$$

If we consider an infinitesimal parallelogram near P, whose sides are the coordinate curves, then the element of area at P is

$$dS = \left| \frac{\partial \mathbf{r}}{\partial u} du \times \frac{\partial \mathbf{r}}{\partial v} dv \right| = \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| du\, dv$$

Therefore, total area of surface is

$$A = \int \int_R \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| du\, dv$$

where R is the region in the uv-plane corresponding to the range of parameter values that define the surface.

Example

Find the element of area on the surface of a sphere of radius a, and hence calculate its total surface area.

Answer

We can represent a point $\mathbf{r}$ on the surface of the sphere in terms of the two parameters θ and ϕ:

$$\mathbf{r}(\theta, \phi) = a \sin\theta \cos\phi \mathbf{i} + a \sin\theta \sin\phi \mathbf{j} + a \cos\theta \mathbf{k}$$

where θ and ϕ are the polar and azimuthal angles respectively. At any point P, vectors tangent to the coordinate curves $\theta = \text{constant}$ and $\phi = \text{constant}$ are

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial \theta} &= a \cos\theta \cos\phi \mathbf{i} + a \cos\theta \sin\phi \mathbf{j} - a \sin\theta \mathbf{k} \\ \frac{\partial \mathbf{r}}{\partial \phi} &= -a \sin\theta \sin\phi \mathbf{i} + a \sin\theta \cos\phi \mathbf{j}\end{aligned}$$

A normal $\mathbf{n}$ to the surface at this point is then given by

$$\begin{aligned}
\mathbf{n} &= \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \phi} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\theta\cos\phi & a\cos\theta\sin\phi & -a\sin\theta \\ -a\sin\theta\sin\phi & a\sin\theta\cos\phi & 0 \end{vmatrix} \\
&= a^2\sin\theta(\sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k})
\end{aligned}$$

which has a magnitude of $a^2\sin\theta$. Therefore the element of area at P is

$$dS = a^2\sin\theta d\theta d\phi$$

and the total surface area of the sphere is given by

$$A = \int_0^{\pi} d\theta \int_0^{2\pi} d\phi\, a^2\sin\theta = 4\pi a^2$$

Vector operators

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

Gradient of a scalar field:

$$\mathrm{grad}\phi = \nabla\phi \equiv \mathbf{i}\frac{\partial \phi}{\partial x} + \mathbf{j}\frac{\partial \phi}{\partial y} + \mathbf{k}\frac{\partial \phi}{\partial z}$$

Now,

$$\begin{aligned}\nabla\phi\cdot d\mathbf{r} &= \left(\mathbf{i}\frac{\partial\phi}{\partial x}+\mathbf{j}\frac{\partial\phi}{\partial y}+\mathbf{k}\frac{\partial\phi}{\partial z}\right)\cdot(\mathbf{i}dx+\mathbf{j}dy+\mathbf{k}dz)\\ &= \frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz\\ &= d\phi\end{aligned}$$

If $\mathbf{r}$ depends on some parameter u such that $\mathbf{r}(u)$ defines a space curve, the total derivative of ϕ with respect to u is

$$\frac{d\phi}{du}=\nabla\phi\cdot\frac{d\mathbf{r}}{du}$$

In general, the rate of change of ϕ with respect to the distance s in a particular direction $\mathbf{a}$ is

$$\frac{d\phi}{ds}=\nabla\phi\cdot\hat{\mathbf{a}}$$

(Directional derivative)

The scalar differential operator $\hat{\mathbf{a}} \cdot \nabla$ gives the rate of change with distance in the direction $\hat{\mathbf{a}}$ of the quantity (vector or scalar) on which it acts. In Cartesian coordinates, it can be written as

$$\hat{\mathbf{a}} \cdot \nabla = a_x \frac{\partial}{\partial x} + a_y \frac{\partial}{\partial y} + a_z \frac{\partial}{\partial z}$$

Thus, we can write the infinitesimal change in an electric field in moving from $\mathbf{r}$ to $\mathbf{r} + d\mathbf{r}$ as $d\mathbf{E} = (d\mathbf{r} \cdot \nabla)\mathbf{E}$.

Consider a surface defined by $\phi(x, y, z) = c$, where c is some constant. If $\hat{\mathbf{t}}$ is a unit tangent to this surface at some point, then $d\phi/ds = 0$ in this direction, and we have $\nabla\phi \cdot \hat{\mathbf{t}} = 0$.

Example

Find expressions for the equations of the tangent plane and line normal to the surface $\phi(x, y, z) = c$ at the point P with coordinates (x_0, y_0, z_0). Use the results to find the equations of the tangent plane and the line normal to the surface of the sphere $\phi = x^2 + y^2 + z^2 = a^2$ at the point $(0, 0, a)$.

Answer

A vector normal to the surface $\phi(x, y, z) = c$ at the point P is $\nabla\phi$, and denoted by $\mathbf{n}_0$. If $\mathbf{r}_0$ is the position vector of the point P relative to the origin, and $\mathbf{r}$ is the position vector of any point on the tangent plane, the vector equation of the tangent plane is

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n}_0 = 0$$

Similarly, the vector equation of the line is

$$(\mathbf{r} - \mathbf{r}_0) \times \mathbf{n}_0 = 0$$

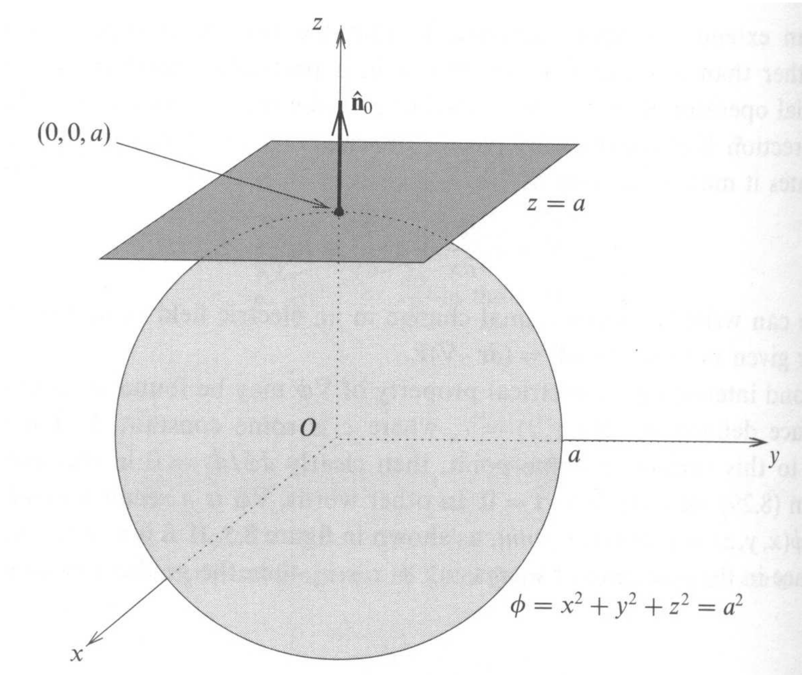

FIG. 3: The tangent plane and the normal to the surface of the sphere $\phi = x^2 + y^2 + z^2 = a^2$ at the point $\mathbf{r}_0$ with coordinates $(0, 0, a)$.

If we now consider the surface of the sphere $\phi = x^2 + y^2 + z^2 = a^2$, then

$$\begin{aligned} \nabla\phi &= 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \\ &= 2a\mathbf{k} \qquad \text{at the point } (0, 0, a) \end{aligned}$$

Therefore, the equation of the tangent plane to the sphere at this point is

$$(\mathbf{r} - \mathbf{r}_0) \cdot 2a\mathbf{k} = 0$$

This gives $2a(z-a) = 0$ or $z = a$. The equation of the line normal to the sphere at the point $(0, 0, a)$ is

$$(\mathbf{r} - \mathbf{r}_0) \times 2a\mathbf{k} = \mathbf{0}$$

which gives $2ay\mathbf{i} - 2ax\mathbf{j} = \mathbf{0}$ or $x = y = 0$ (z-axis).

Divergence of a vector field

The divergence of a vector field $\mathbf{a}(x, y, z)$ is defined as

$$\operatorname{div}\mathbf{a} = \nabla \cdot \mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

Curl of a vector field

The curl of a vector field $\mathbf{a}(x, y, z)$ is defined as

$$\begin{aligned}
\operatorname{curl}\mathbf{a} &= \nabla \times \mathbf{a} \\
&= \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\mathbf{j} \\
&\quad + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\mathbf{k} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}
\end{aligned}$$

Vector operators acting on sums and products

$$
\begin{aligned}
\nabla(\phi+\psi) &= \nabla\phi+\nabla\psi \\
\nabla\cdot(\mathbf{a}+\mathbf{b}) &= \nabla\cdot\mathbf{a}+\nabla\cdot\mathbf{b} \\
\nabla\times(\mathbf{a}+\mathbf{b}) &= \nabla\times\mathbf{a}+\nabla\times\mathbf{b} \\
\nabla(\phi\psi) &= \phi\nabla\psi+\psi\nabla\phi \\
\nabla(\mathbf{a}\cdot\mathbf{b}) &= \mathbf{a}\times(\nabla\times\mathbf{b})+\mathbf{b}\times(\nabla\times\mathbf{a}) \\
&\quad +(\mathbf{a}\cdot\nabla)\mathbf{b}+(\mathbf{b}\cdot\nabla)\mathbf{a} \\
\nabla\cdot(\phi\mathbf{a}) &= \phi\nabla\cdot\mathbf{a}+\mathbf{a}\cdot\nabla\phi \\
\nabla\cdot(\mathbf{a}\times\mathbf{b}) &= \mathbf{b}\cdot(\nabla\times\mathbf{a})-\mathbf{a}\cdot(\nabla\times\mathbf{b}) \\
\nabla\times(\phi\mathbf{a}) &= \nabla\phi\times\mathbf{a}+\phi\nabla\times\mathbf{a} \\
\nabla\times(\mathbf{a}\times\mathbf{b}) &= \mathbf{a}(\nabla\cdot\mathbf{b})-\mathbf{b}(\nabla\cdot\mathbf{a})+(\mathbf{b}\cdot\nabla)\mathbf{a} \\
&\quad -(\mathbf{a}\cdot\nabla)\mathbf{b}
\end{aligned}
$$

Some useful special cases

$$
\begin{aligned}
\nabla\phi(r) &= \frac{d\phi}{dr}\hat{\mathbf{r}} \\
\nabla\cdot[\phi(r)\mathbf{r}] &= 3\phi(r) + r\frac{d\phi(r)}{dr} \\
\nabla^2\phi(r) &= \frac{d^2\phi(r)}{dr^2} + \frac{2}{r}\frac{d\phi(r)}{dr} \\
\nabla\times[\phi(r)\mathbf{r}] &= \mathbf{0}
\end{aligned}
$$

where $r = |\mathbf{r}|$.

We also have

$$
\begin{aligned}
\nabla\mathbf{r} &= \hat{\mathbf{r}} \\
\nabla\cdot\mathbf{r} &= 3 \\
\nabla\times\mathbf{r} &= \mathbf{0}
\end{aligned}
$$

Combinations of grad, div and curl

$$
\begin{aligned}
\nabla \times \nabla\phi &= \mathbf{0} \\
\nabla \cdot (\nabla \times \mathbf{a}) &= 0 \\
\nabla \cdot \nabla\phi &= \nabla^2\phi \\
\nabla(\nabla \cdot \mathbf{a}) &= \left(\frac{\partial^2 a_x}{\partial x^2} + \frac{\partial^2 a_y}{\partial x \partial y} + \frac{\partial^2 a_z}{\partial x \partial z}\right)\mathbf{i} \\
&\quad \left(\frac{\partial^2 a_x}{\partial y \partial x} + \frac{\partial^2 a_y}{\partial y^2} + \frac{\partial^2 a_z}{\partial y \partial z}\right)\mathbf{j} \\
&\quad \left(\frac{\partial^2 a_x}{\partial z \partial x} + \frac{\partial^2 a_y}{\partial z \partial y} + \frac{\partial^2 a_z}{\partial z^2}\right)\mathbf{k} \\
\nabla \times (\nabla \times \mathbf{a}) &= \nabla(\nabla \cdot \mathbf{a}) - \nabla^2\mathbf{a}
\end{aligned}
$$

Example

Show that $\nabla \cdot (\nabla\phi \times \nabla\psi) = 0$ where ϕ and ψ are scalar fields.

Answer

We have

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b}).$$

If we let $\mathbf{a} = \nabla\phi$ and $\mathbf{b} = \nabla\psi$, we obtain

$$\nabla \cdot (\nabla\phi \times \nabla\psi) = \nabla\psi \cdot (\nabla \times \nabla\phi) - \nabla\phi \cdot (\nabla \times \nabla\psi) = 0$$

since $\nabla \times \nabla\phi = 0 = \nabla \times \nabla\psi$.

Cylindrical polar coordinates

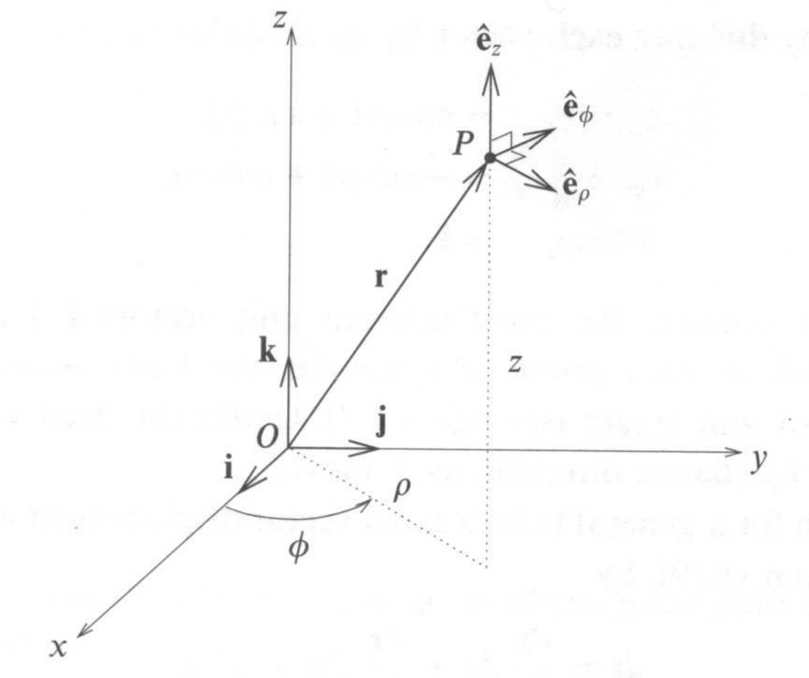

FIG. 4: Cylindrical polar coordinates ρ, ϕ, z

The position of a point P having Cartesian coordinates x, y, z may be expressed in terms of cylindrical polar coordinates ρ, ϕ, z where

$$x = \rho \cos\phi, \quad y = \rho \sin\phi, \quad z = z \tag{11}$$

and $\rho \geq 0$, $0 \leq \phi < 2\pi$ and $-\infty < z < \infty$. The position vector of P may be written as

$$\mathbf{r} = \rho \cos\phi \mathbf{i} + \rho \sin\phi \mathbf{j} + z\mathbf{k}$$

Taking partial derivatives with respect to ρ, ϕ, z respectively, we obtain

$$\begin{aligned}
\mathbf{e}_\rho &= \frac{\partial \mathbf{r}}{\partial \rho} &&= \cos\phi\mathbf{i} + \sin\phi\mathbf{j} \\
\mathbf{e}_\phi &= \frac{\partial \mathbf{r}}{\partial \phi} &&= -\rho\sin\phi\mathbf{i} + \rho\cos\phi\mathbf{j} \\
\mathbf{e}_z &= \frac{\partial \mathbf{r}}{\partial z} &&= \mathbf{k}
\end{aligned}$$

These vectors lie in the direction of increasing ρ, ϕ and z respectively, but are not all of unit length.

The unit vectors are

$$\begin{aligned}
\hat{\mathbf{e}}_\rho &= \mathbf{e}_\rho &&= \cos\phi\mathbf{i} + \sin\phi\mathbf{j} \\
\hat{\mathbf{e}}_\phi &= \tfrac{1}{\rho}\mathbf{e}_\phi &&= -\sin\phi\mathbf{i} + \cos\phi\mathbf{j} \\
\hat{\mathbf{e}}_z &= \mathbf{e}_z &&= \mathbf{k}
\end{aligned}$$

The expression for a general infinitesimal vector displacement $d\mathbf{r}$ in the position of P is given by

$$\begin{aligned} d\mathbf{r} &= \frac{\partial \mathbf{r}}{\partial \rho} d\rho + \frac{\partial \mathbf{r}}{\partial \phi} d\phi + \frac{\partial \mathbf{r}}{\partial z} dz \\ &= d\rho\, \mathbf{e}_\rho + d\phi\, \mathbf{e}_\phi + dz\, \mathbf{e}_z \\ &= d\rho\, \hat{\mathbf{e}}_\rho + \rho\, d\phi\, \hat{\mathbf{e}}_\phi + dz\, \hat{\mathbf{e}}_z \end{aligned}$$

The magnitude ds of the displacement $d\mathbf{r}$ is given in cylindrical polar coordinates by

$$(ds)^2 = d\mathbf{r} \cdot d\mathbf{r} = (d\rho)^2 + \rho^2 (d\phi)^2 + (dz)^2$$

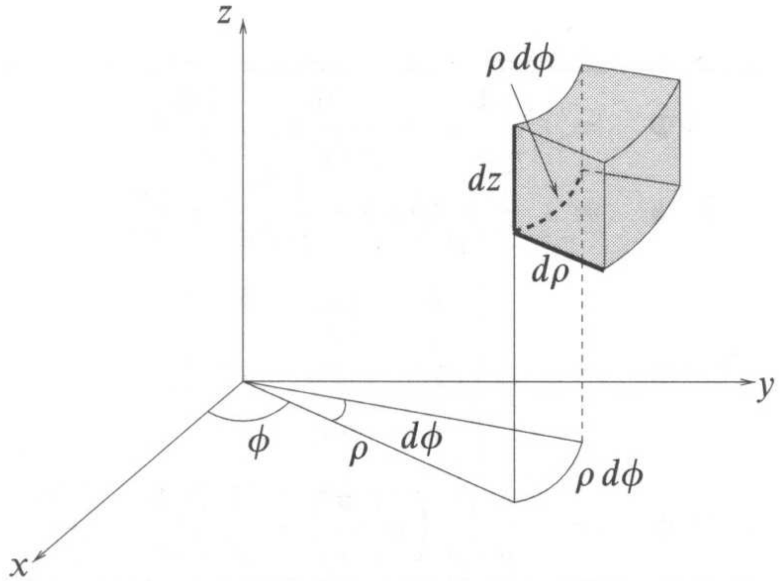

FIG. 5: The element of volume in cylindrical polar coordinates is given by $\rho\, d\rho\, d\phi\, dz$.

Volume of the infinitesimal parallelepiped defined by the vectors $d\rho\,\hat{\mathbf{e}}_\rho$, $\rho\, d\phi\,\hat{\mathbf{e}}_\phi$ and $dz\,\hat{\mathbf{e}}_z$ is given by

$$dV = |d\rho\,\hat{\mathbf{e}}_\rho \cdot (\rho\, d\phi\,\hat{\mathbf{e}}_\phi \times dz\,\hat{\mathbf{e}}_z)| = \rho\, d\rho\, d\phi\, dz$$

since the basis vectors are orthonormal.

Example

Express the vector field $\mathbf{a} = yz\mathbf{i} - y\mathbf{j} + xz^2\mathbf{k}$ in cylindrical polar coordinates, and hence calculate its divergence. Show that the same result is obtained by evaluating the divergence in Cartesian coordinates.

Answer

From the basis vectors of the cylindrical polar coordinate, we obtain

$$\begin{aligned}
\mathbf{i} &= \cos\phi\,\hat{\mathbf{e}}_\rho - \sin\phi\,\hat{\mathbf{e}}_\phi \\
\mathbf{j} &= \sin\phi\,\hat{\mathbf{e}}_\rho + \cos\phi\,\hat{\mathbf{e}}_\phi \\
\mathbf{k} &= \hat{\mathbf{e}}_z
\end{aligned}$$

Substituting these relations and (11) into the expression for $\mathbf{a}$ we find

$$\begin{aligned}
\mathbf{a} &= z\rho\sin\phi(\cos\phi\,\hat{\mathbf{e}}_\rho - \sin\phi\,\hat{\mathbf{e}}_\phi) \\
&\quad -\rho\sin\phi(\sin\phi\,\hat{\mathbf{e}}_\rho + \cos\phi\,\hat{\mathbf{e}}_\phi) + z^2\rho\cos\phi\,\hat{\mathbf{e}}_z \\
&= (z\rho\sin\phi\cos\phi - \rho\sin^2\phi)\hat{\mathbf{e}}_\rho \\
&\quad -(z\rho\sin^2\phi + \rho\sin\phi\cos\phi)\hat{\mathbf{e}}_\phi + z^2\rho\cos\phi\,\hat{\mathbf{e}}_z
\end{aligned}$$

Substituting into the expression for $\nabla \cdot \mathbf{a}$, we have

$$\begin{aligned}
\nabla \cdot \mathbf{a} &= 2z\sin\phi - 2\sin^2\phi - 2z\sin\phi\cos\phi \\
&\quad - \cos^2\phi + \sin^2\phi + 2z\rho\cos\phi \\
&= 2z\rho\cos\phi - 1.
\end{aligned}$$

Calculating the divergence directly in Cartesian coordinates, we have

$$\nabla \cdot \mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} = 2zx - 1 = 2z\rho\cos\phi - 1$$

Spherical polar coordinates

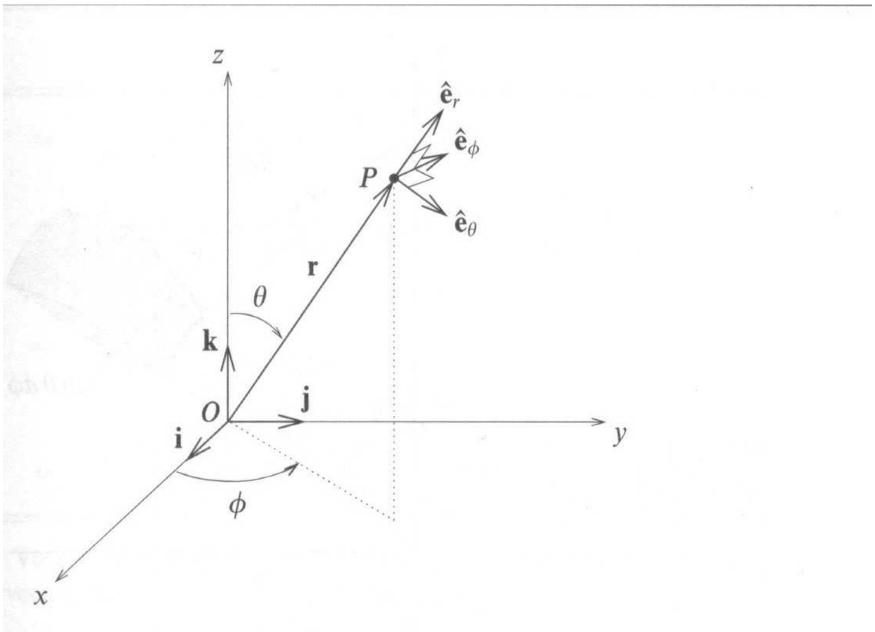

FIG. 6: Spherical polar coordinates ρ, θ, ϕ

The position of a point P with Cartesian coordinates x, y and z may be expressed in terms of spherical polar coordinates r, θ and ϕ, where

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

and $r \geq 0$, $0 \leq \theta \leq \pi$, and $0 \leq \phi < 2\pi$. The position vector P is

$$\mathbf{r} = r\sin\theta\cos\phi\,\mathbf{i} + r\sin\theta\sin\phi\,\mathbf{j} + r\cos\theta\,\mathbf{k}$$

The unit basis vectors are

$$\begin{aligned}
\hat{\mathbf{e}}_r &= \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\mathbf{k} \\
\hat{\mathbf{e}}_\theta &= \cos\theta\cos\phi\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} - \sin\theta\,\mathbf{k} \\
\hat{\mathbf{e}}_\phi &= -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}
\end{aligned}$$

A general infinitesimal vector displacement in spherical polars is

$$d\mathbf{r} = dr\,\hat{\mathbf{e}}_r + r\,d\theta\,\hat{\mathbf{e}}_\theta + r\,\sin\theta\,d\phi\,\hat{\mathbf{e}}_\phi$$

The magnitude ds of the displacement $d\mathbf{r}$ is given by

$$(ds)^2 = d\mathbf{r}\cdot d\mathbf{r} = (dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta(d\phi)^2$$

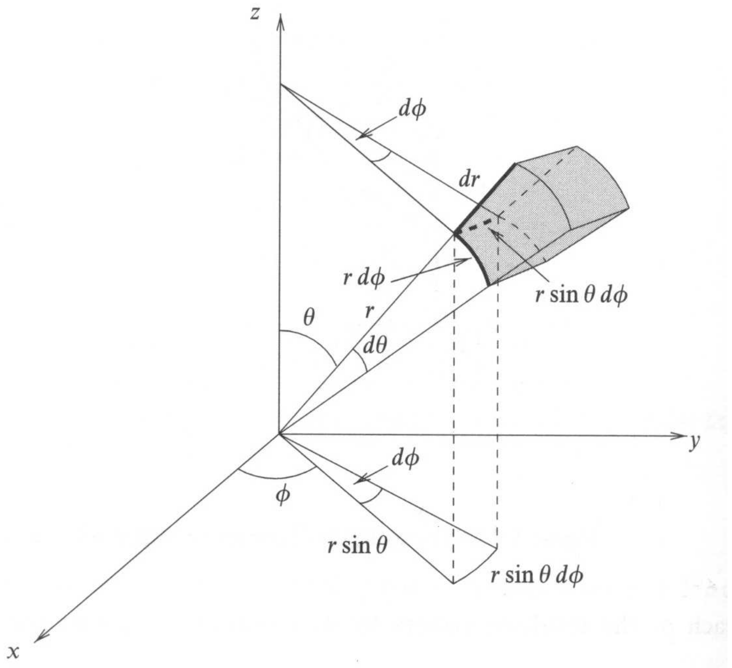

FIG. 7: The element of volume in spherical polar coordinates is given by $r^2 \sin\theta\, dr\, d\theta\, d\phi$.

The volume of the infinitesimal parallelepiped defined by the vectors $dr\,\hat{\mathbf{e}}_r$, $r\,d\theta\,\hat{\mathbf{e}}_\theta$, and $r\sin\theta\,d\phi\,\hat{\mathbf{e}}_\phi$ is given by

$$dV = |dr\,\hat{\mathbf{e}}_r \cdot (r\,d\theta\,\hat{\mathbf{e}}_\theta \times r\sin\theta\,d\phi\,\hat{\mathbf{e}}_\phi)| = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$
\begin{aligned}
\nabla\Phi &= \frac{\partial\Phi}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\mathbf{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\hat{\mathbf{e}}_\phi \\
\nabla\cdot\mathbf{a} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 a_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, a_\theta) \\
&\quad + \frac{1}{r\sin\theta}\frac{\partial a_\phi}{\partial\phi} \\
\nabla\times\mathbf{a} &= \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\mathbf{e}}_r & r\hat{\mathbf{e}}_\theta & r\sin\theta\,\hat{\mathbf{e}}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ a_r & r a_\theta & r\sin\theta\, a_\phi \end{vmatrix} \\
\nabla^2\Phi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) \\
&\quad + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}
\end{aligned}
$$