PC2174

Tutorial 2: Line, Surface and Volume Integrals

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1. $\mathbf{F}$ is a vector $xy^2\mathbf{i} + 2\mathbf{j} + x\mathbf{k}$, and L a path parametrised by $x = ct$, $y = c/t$, $z = d$ for the range $1 \le t \le 2$. Evaluate (a) $\int_L \mathbf{F}\, dt$, (b) $\int_L \mathbf{F}\, dy$, and (c) $\int_L \mathbf{F} \cdot d\mathbf{r}$.

2. A single-turn coil C of arbitrary shape is placed in a magnetic field $\mathbf{B}$ and carries a current I. Show that the couple acting upon the coil can be written as

$$\mathbf{M} = I \int_C (\mathbf{B} \cdot \mathbf{r}) d\mathbf{r} - I \int_C \mathbf{B}(\mathbf{r} \cdot d\mathbf{r}).$$

   For a planar rectangular coil of sides $2a$ and $2b$ placed with its plane vertical and at an angle ϕ to a uniform horizontal field $\mathbf{B}$, show that $\mathbf{M}$ is, as expected, $4abBI\cos\phi\,\mathbf{k}$.

3. Show that the expression below is equal to the solid angle subtended by a rectangular aperture of sides $2a$ and $2b$ at a point a distance c from the aperture along the normal to its centre:

$$\Omega = 4 \int_0^b \frac{ac}{(y^2 + c^2)(y^2 + c^2 + a^2)^{1/2}} dy.$$

   By setting $y = (a^2 + c^2)^{1/2} \tan\phi$, change this integral into the form

$$\int_0^{\phi_1} \frac{4ac\cos\phi}{c^2 + a^2 \sin^2\phi} d\phi,$$

   where $\tan\phi_1 = b/(a^2 + c^2)^{1/2}$, and hence show that

$$\Omega = 4 \tan^{-1} \left[\frac{ab}{c(a^2 + b^2 + c^2)^{1/2}} \right].$$

4. A vector field $\mathbf{a} = -zxr^{-3}\mathbf{i} - zyr^{-3}\mathbf{j} + (x^2 + y^2)r^{-3}\mathbf{k}$, where $r^2 = x^2+y^2+z^2$. Show that the field is conservative (a) by showing $\nabla\times\mathbf{a} = \mathbf{0}$, and (b) by constructing its potential function ϕ.

5. The vector field $\mathbf{a} = (z^2 + 2xy)\mathbf{i} + (x^2 + 2yz)\mathbf{j} + (y^2 + 2zx)\mathbf{k}$. Show that $\mathbf{a}$ is conservative and that the line integral $\int \mathbf{a} \cdot d\mathbf{r}$ along any line joining $(1, 1, 1)$ and $(1, 2, 2)$ has the value 11.