Line, Surface and Volume Integrals

Line integrals

$$\int_C \phi \, d\mathbf{r}, \quad \int_C \mathbf{a} \cdot d\mathbf{r}, \quad \int_C \mathbf{a} \times d\mathbf{r} \qquad (1)$$

(ϕ is a scalar field and $\mathbf{a}$ is a vector field)

We divide the path C joining the points A and B into N small line elements $\Delta\mathbf{r}_p$, $p = 1, \ldots, N$. If (x_p, y_p, z_p) is any point on the line element Δr_p, then the second type of line integral in Eq. (1) is defined as

$$\int_C \mathbf{a} \cdot d\mathbf{r} = \lim_{N \to \infty} \sum_{p=1}^{N} \mathbf{a}(x_p, y_p, z_p) \cdot \mathbf{r}_p$$

where it is assumed that all $|\Delta\mathbf{r}_p| \to 0$ as $N \to \infty$.

Evaluating line integrals

The first type of line integral in Eq. (1) can be written as

$$\int_C \phi\, d\mathbf{r} = \mathbf{i}\int_C \phi(x,y,z)\,dx + \mathbf{j}\int_C \phi(x,y,z)\,dy + \mathbf{k}\int_C \phi(x,y,z)\,dz$$

The three integrals on the RHS are ordinary scalar integrals.

The second and third line integrals in Eq. (1) can also be reduced to a set of scalar integrals by writing the vector field $\mathbf{a}$ in terms of its Cartesian components as $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$. Thus,

$$\begin{aligned}
\int_C \mathbf{a}\cdot d\mathbf{r} &= \int_C (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k})\cdot(dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}) \\
&= \int_C (a_x dx + a_y dy + a_z dz) \\
&= \int_C a_x dx + \int_C a_y dy + \int_C a_z dz
\end{aligned}$$

Some useful properties about line integrals:

1. Reversing the path of integration changes the sign of the integral. That is,

$$\int_A^B \mathbf{a} \cdot d\mathbf{r} = -\int_B^A \mathbf{a} \cdot d\mathbf{r}$$

2. If the path of integration is subdivided into smaller segments, then the sum of the separate line integrals along each segment is equal to the line integral along the whole path. That is,

$$\int_A^B \mathbf{a} \cdot d\mathbf{r} = \int_A^P \mathbf{a} \cdot d\mathbf{r} + \int_P^B \mathbf{a} \cdot d\mathbf{r}$$

Example

Evaluate the line integral $I = \int_C \mathbf{a} \cdot d\mathbf{r}$, where $\mathbf{a} = (x+y)\mathbf{i} + (y-x)\mathbf{j}$, along each of the paths in the xy-plane shown in the figure below, namely,

1. the parabola $y^2 = x$ from $(1,1)$ to $(4,2)$,
2. the curve $x = 2u^2 + u + 1$, $y = 1 + u^2$ from $(1,1)$ to $(4,2)$,
3. the line $y = 1$ from $(1,1)$ to $(4,1)$, followed by the line $y = x$ from $(4,1)$ to $(4,2)$.

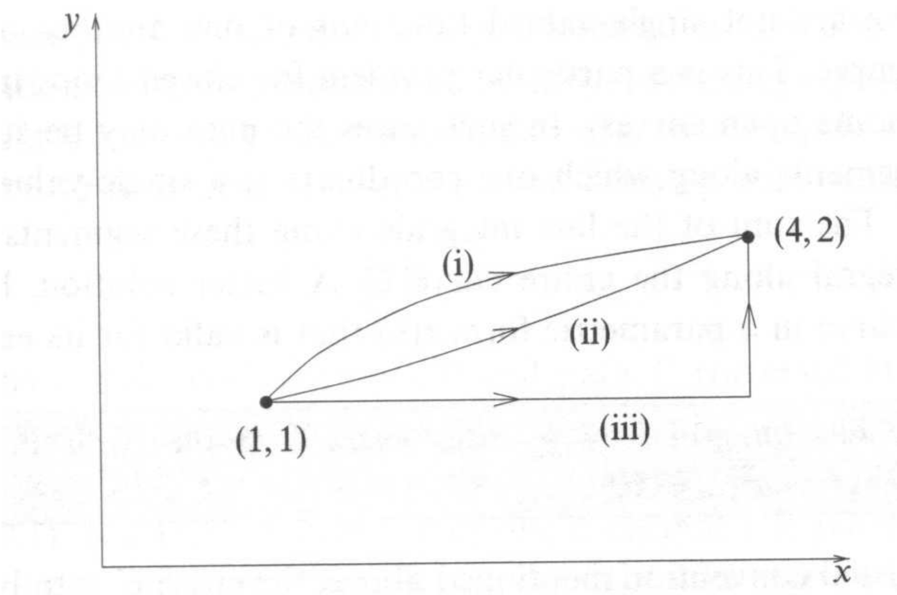

FIG. 1: Different possible paths between points $(1,1)$ and $(4,2)$.

Answer

Since each of the path lies entirely in the xy-plane, we have $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$. Therefore,

$$I = \int_C \mathbf{a} \cdot d\mathbf{r} = \int_C [(x+y)dx + (y-x)dy]. \quad (2)$$

We now evaluate the line integral along each path.

Case (i). Along the parabola $y^2 = x$ we have $2y\,dy = dx$. Substituting for x in Eq. (2) and using just the limits on y, we obtain

$$\begin{aligned} I &= \int_{(1,1)}^{(4,2)} [(x+y)dx + (y-x)dy] \\ &= \int_1^2 [(y^2+y)2y + (y-y^2)]dy = 11\frac{1}{3} \end{aligned}$$

Case (ii). The second path is given in terms of parameter u. We could eliminate u between two equations to obtain a relationship between x and y directly, and proceed as above, but it us usually quicker to write the line integral in terms of parameter u. Along the curve $x = 2u^2 + u + 1$, $y = 1 + u^2$, we have $dx = (4u + 1)\,du$ and $dy = 2u\,du$. Substituting for x and y in Eq. (2) and writing the correct limits on u, we obtain

$$
\begin{aligned}
I &= \int_{(1,1)}^{(4,2)} [(x+y)dx + (y-x)dy] \\
&= \int_0^1 [(3u^2 + u + 2)(4u+1) - (u^2+u)2u]du \\
&= 10\frac{2}{3}
\end{aligned}
$$

Case (iii). For the third path the line integral must be evaluated along the two line segments separately and the results added together. First, along the line $y = 1$, we have $dy = 0$. Substituting this into Eq. (2) and using just the limits on x for this segment, we obtain

$$\begin{aligned}\int_{(1,1)}^{(4,1)} [(x+y)dx + (y-x)dy] &= \int_1^4 (x+1)dx \\ &= 10\frac{1}{2}\end{aligned}$$

Along the line $x = 4$, we have $dx = 0$. Substituting this into Eq. (2) and using just the limits on y, we obtain

$$\begin{aligned}\int_{(4,1)}^{(4,2)} [(x+y)dx + (y-x)dy] &= \int_1^2 (y-4)dy \\ &= -2\frac{1}{2}\end{aligned}$$

Therefore, the value of the line integral along the whole path is $10\frac{1}{2} - 2\frac{1}{2} = 8$.

Connectivity of regions

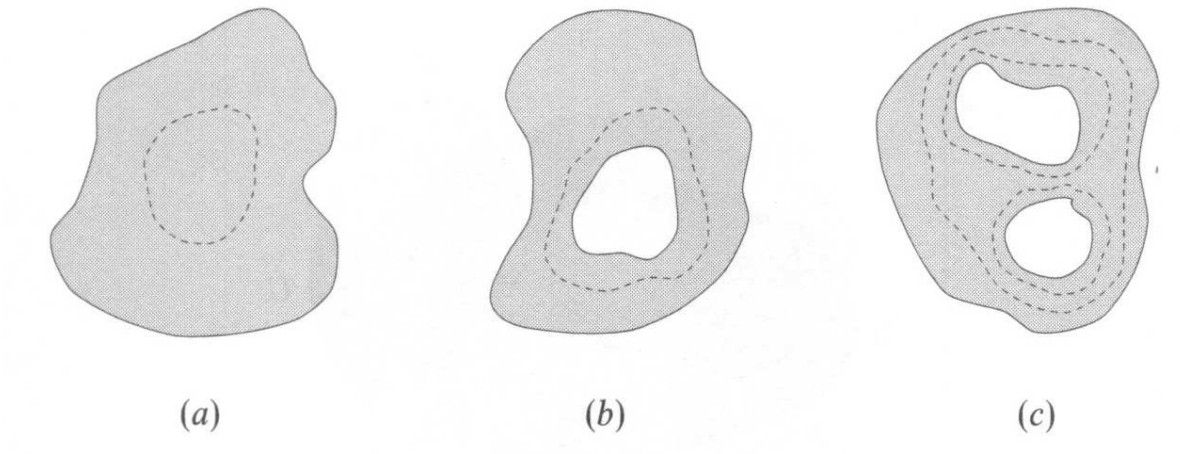

FIG. 2: (a) A simply connected region; (b) a doubly connected region; (c) a triply connected region.

A plane region R is simply connected if any closed curve within R can be continuously shrunk to a point without leaving the region. If, however, the region R contains a hole then there exits simple closed curves that cannot be shrunk to a point without leaving R. Such a region is doubly connected. Similarly, a region with $n-1$ holes is said to be n-fold connected, or multiply connected.

Green's theorem in a plane

Suppose the functions $P(x, y)$, $Q(x, y)$ and their partial derivatives are single-valued, finite and continuous inside and on the boundary C of some simply connected region R in the xy-plane. Green's theorem in a plane then states that

$$\oint_C (P\,dx + Q\,dy) = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy \quad (3)$$

To prove this, let us consider the simply connected region R below.

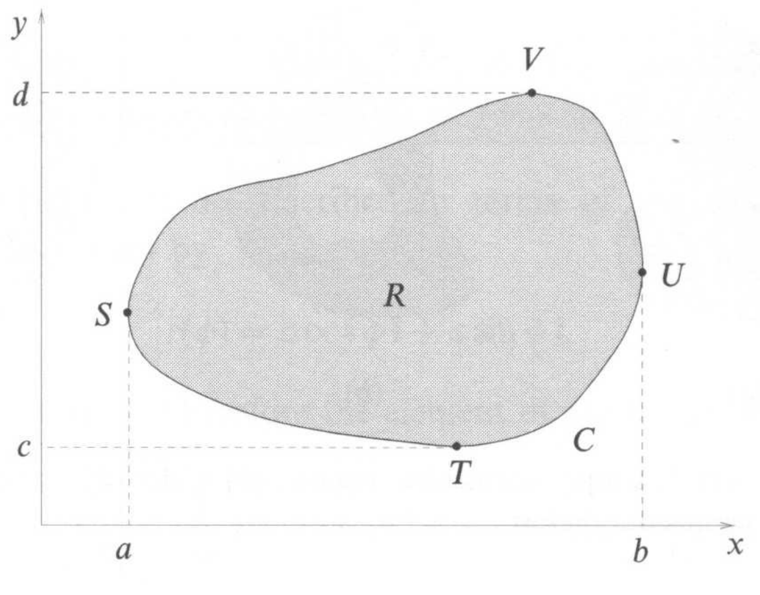

FIG. 3: A simply connected region R bounded by the curve C.

Let $y = y_1(x)$ and $y = y_2(x)$ be the equations of the curves STU and SVU respectively. We then write

$$\begin{aligned}
\int\int_R \frac{\partial P}{\partial y} &= \int_a^b dx \int_{y_1(x)}^{y_2(x)} dy \, \frac{\partial P}{\partial y} \\
&= \int_a^b dx \, [P(x,y)]_{y=y_1(x)}^{y=y_2(x)} \\
&= \int_a^b \left[P(x, y_2(x)) - P(x, y_1(x)) \right] dx \\
&= -\int_a^b P(x, y_1(x)) dx - \int_a^b P(x, y_2(x)) dx \\
&= -\oint_C P \, dx
\end{aligned}$$

If we now let $x = x_1(y)$ and $x = x_2(y)$ be the equations of the curves TSV and TUV respectively, we can similarly show that

$$\begin{aligned}
\int\int_R \frac{\partial Q}{\partial x} dx\, dy &= \int_c^d dy \int_{x_1(y)}^{x_2(y)} dx \frac{\partial Q}{\partial x} \\
&= \int_c^d dy\, [Q(x,y)]_{x=x_1(y)}^{x=x_2(y)} \\
&= \int_c^d [Q(x_2(y), y) - Q(x_1(y), y)]\, dy \\
&= \int_c^d Q(x_1, y) dy + \int_c^d Q(x_2, y) dy \\
&= \oint_C Q dy
\end{aligned}$$

Subtracting these two results gives Green's theorem in a plane.

Example

Show that the area of a region R enclosed by a simple closed curve C is given by
$A = \frac{1}{2}\oint_C (x\,dy - y\,dx) = \oint_C x\,dy = -\oint_C y\,dx$.
Hence, calculate the area of the ellipse $x = a\cos\phi$, $y = b\sin\phi$.

Answer

In Green's theorem, put $P = -y$ and $Q = x$. Then

$$\oint_C (x\,dy - y\,dx) = \int\int_R (1+1) dx\,dy = 2\int\int_R dx\,dy = 2A$$

Therefore, the area of the region is
$A = \frac{1}{2}\oint_C (xdy - ydx)$.

Alternatively, we could put $P = 0$ and $Q = x$ and obtain $A = \oint_C xdy$, or put $P = -y$ and $Q = 0$, which gives $A = -\oint_C ydx$.

The area of the ellipse $x = a\cos\phi$, $y = b\sin\phi$ is given by

$$\begin{aligned} A &= \frac{1}{2}\oint_C (xdy - ydx) \\ &= \frac{1}{2}\int_0^{2\pi} ab(\cos^2\phi + \sin^2\phi)\, d\phi \\ &= \frac{ab}{2}\int_0^{2\pi} d\phi = \pi ab. \end{aligned}$$

Conservative fields and potentials

For line integrals of the form $\int_C \mathbf{a} \cdot d\mathbf{r}$, there exists a class of vector fields for which the line integral between two points is independent of the path taken. Such vector fields are called conservative.

A vector field $\mathbf{a}$ that has continuous partial derivatives in a simply connected region R is conservative if, and only if, any of the following is true.

1. The integral $\int_A^B \mathbf{a} \cdot d\mathbf{r}$, where A and B lie in the region R, is independent of the path from A to B. Hence the integral $\oint_C \mathbf{a} \cdot d\mathbf{r}$ around any closed loop in R is zero.
2. There exits a single-valued function ϕ of position such that $\mathbf{a} = \nabla \phi$.
3. $\nabla \times \mathbf{a} = 0$
4. $\mathbf{a} \cdot d\mathbf{r}$ is an exact differential.

If the line integral from A to B is independent of the path taken between the points, then its value must be a function only of the positions of A and B. We write

$$\int_A^B \mathbf{a} \cdot d\mathbf{r} = \phi(B) - \phi(A) \tag{4}$$

which defines a single-valued scalar function of position ϕ. If the points A and B are separated by an infinitesimal displacement $d\mathbf{r}$, then Eq. (4) becomes

$$\mathbf{a} \cdot d\mathbf{r} = d\phi$$

which requires $\mathbf{a} \cdot d\mathbf{r}$ to be an exact differential. But $d\phi = \nabla\phi \cdot d\mathbf{r}$, so

$$(\mathbf{a} - \nabla\phi) \cdot d\mathbf{r} = 0.$$

Since $d\mathbf{r}$ is arbitrary, $\mathbf{a} = \nabla\phi$, which implies $\nabla \times \mathbf{a} = 0$.

Example

Evaluate the line integral $I = \int_A^B \mathbf{a} \cdot d\mathbf{r}$, where $\mathbf{a} = (xy^2 + z)\mathbf{i} + (x^2 y + 2)\mathbf{j} + x\mathbf{k}$, A is the point (c, c, h) and B is the point $(2c, c/2, h)$, along the different paths

1. C_1, given by $x = cu$, $y = c/u$, $z = h$, and
2. C_2, given by $2y = 3c - x$, $z = h$.

Show that the vector field $\mathbf{a}$ is in fact conservative, and find ϕ such that $\mathbf{a} = \nabla\phi$.

Answer

Expanding out the integrand, we have

$$I = \int_{(c,c,h)}^{(2c,c/2,h)} \left[(xy^2 + z)dx + (x^2y + 2)dy + xdz\right] \tag{5}$$

(i). Along C_1, we have $dx = cdu$, $dy = -(c/u^2)\,du$, and on substituting in Eq. (5) and finding the limits on u, we obtain

$$I = \int_1^2 c\left(h - \frac{2}{u^2}\right) du = c(h-1)$$

(ii) Along C_2, we have $2dy = -dx$, $dz = 0$, and on substituting in Eq. (5) and using the limits on x, we obtain

$$I = \int_c^{2c} \left(\frac{1}{2}x^3 - \frac{9}{4}cx^2 + \frac{9}{4}c^2x + h - 1\right) dx = c(h-1).$$

Hence the line integral has the same value along both paths. Taking the curl of $\mathbf{a}$, we have

$$\nabla \times \mathbf{a} = (0-0)\mathbf{i} + (1-1)\mathbf{j} + (2xy - 2xy)\mathbf{k} = \mathbf{0}.$$

$\mathbf{a}$ is a conservative field.

Thus, we can write $\mathbf{a} = \nabla\phi$. Therefore, ϕ must satisfy

$$\frac{\partial\phi}{\partial x} = xy^2 + z$$

which implies that $\phi = \frac{1}{2}x^2y^2 + zx + f(y,z)$ for some function f. Secondly, we require

$$\frac{\partial\phi}{\partial y} = x^2y + \frac{\partial f}{\partial y} = x^2y + 2$$

which implies $f = 2y + g(z)$. Finally, since

$$\frac{\partial\phi}{\partial z} = x + \frac{\partial g}{\partial z} = x$$

we have $g = \text{constant} = k$. So we have constructed the function $\phi = \frac{1}{2}x^2y^2 + zx + 2y + k$.

Surface integrals

Examples,

$$\int_S \phi \, dS, \quad \int_S \phi \, d\mathbf{S}, \quad \int_S \mathbf{a} \cdot d\mathbf{S}, \quad \int_S \mathbf{a} \times d\mathbf{S}$$

S may be either open or close. The integrals, in general, are double integrals.

The vector differential $d\mathbf{S}$ represents a vector area element of the surface S, and may be written as $d\mathbf{S} = \hat{\mathbf{n}} \, dS$, where $\hat{\mathbf{n}}$ is a unit normal to the surface at the position of the element.

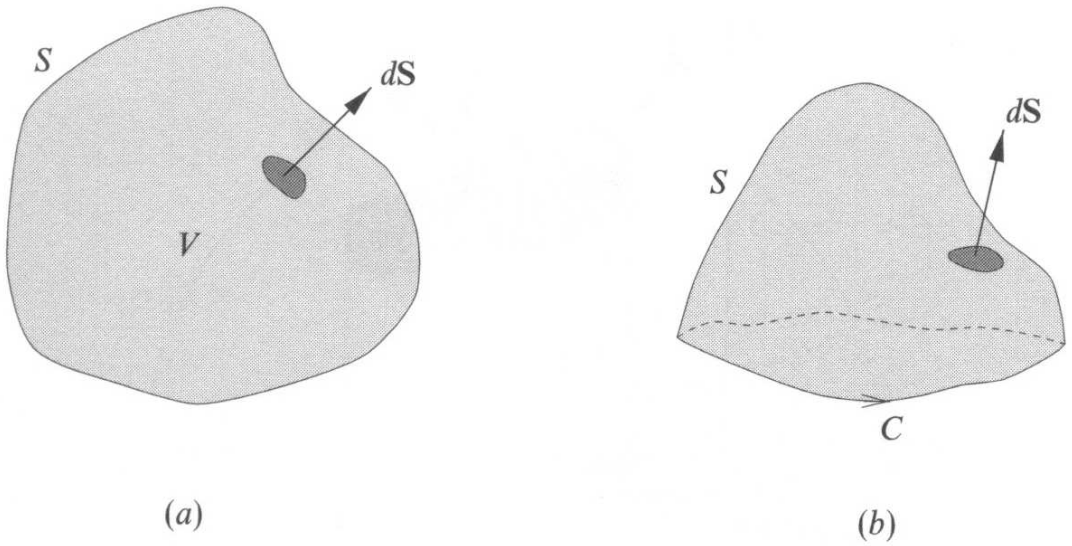

FIG. 4: (a) A closed surface and (b) an open surface. In each case a normal to the surface is shown: $d\mathbf{S} = \hat{\mathbf{n}} \, dS$.

The formal definition of a surface integral: We divide the surface S into N elements of area ΔS_p, $p = 1, \ldots, N$, each with a unit normal $\hat{\mathbf{n}}_p$. If (x_p, y_p, z_p) is any point in ΔS_p, then

$$\int_S \mathbf{a} \cdot d\mathbf{S} = \lim_{N \to \infty} \sum_{p=1}^{N} \mathbf{a}(x_p, y_p, z_p) \cdot \hat{\mathbf{n}}_p \Delta S_p$$

where it is required that all $\Delta S_p \to 0$ as $N \to \infty$.

Evaluating surface integrals

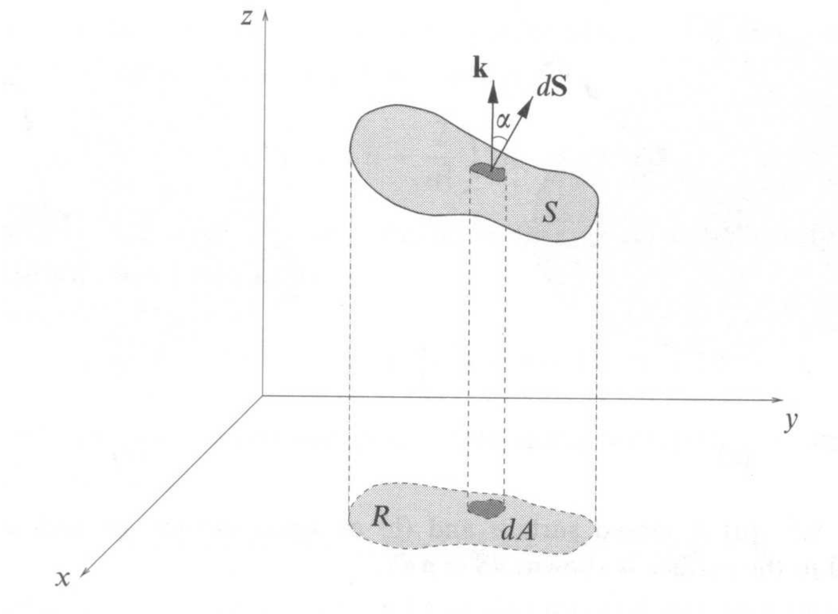

FIG. 5: A surface S (or part thereof) projected onto a region R in the xy-plane; $d\mathbf{S}$ is the surface element at a point P.

The surface S is projected onto a region R of the xy-plane, so that an element of surface area dS at point P projects onto the area element dA. We see that $dA = |\cos\alpha| dS$, where α is the angle between the unit vector $\mathbf{k}$ in the z-direction and the unit normal $\hat{\mathbf{n}}$ to the surface at P.

So, at any given point of S, we have simply

$$dS = \frac{dA}{|\cos\alpha|} = \frac{dA}{|\hat{\mathbf{n}}\cdot\mathbf{k}|}.$$

Now, if the surface S is given by the equation $f(x, y, z) = 0$, then the unit normal at any point of the surface is simply given by $\hat{\mathbf{n}} = \nabla f/|\nabla f|$ evaluated at that point. The scalar element of surface area then becomes

$$dS = \frac{dA}{|\hat{\mathbf{n}}\cdot\mathbf{k}|} = \frac{|\nabla f|\,dA}{\nabla f\cdot\mathbf{k}} = \frac{|\nabla f|\,dA}{\partial f/\partial z} \tag{6}$$

where $|\nabla f|$ and $\partial f/\partial z$ are evaluated on the surface S. We can therefore express any surface integral over S as a double integral over the region R in the xy-plane.

Example

Evaluate the surface integral $I = \int_S \mathbf{a} \cdot d\mathbf{S}$, where $\mathbf{a} = x\mathbf{i}$ and S is the surface of the hemisphere $x^2 + y^2 + z^2 = a^2$ with $z \geq 0$.

Answer

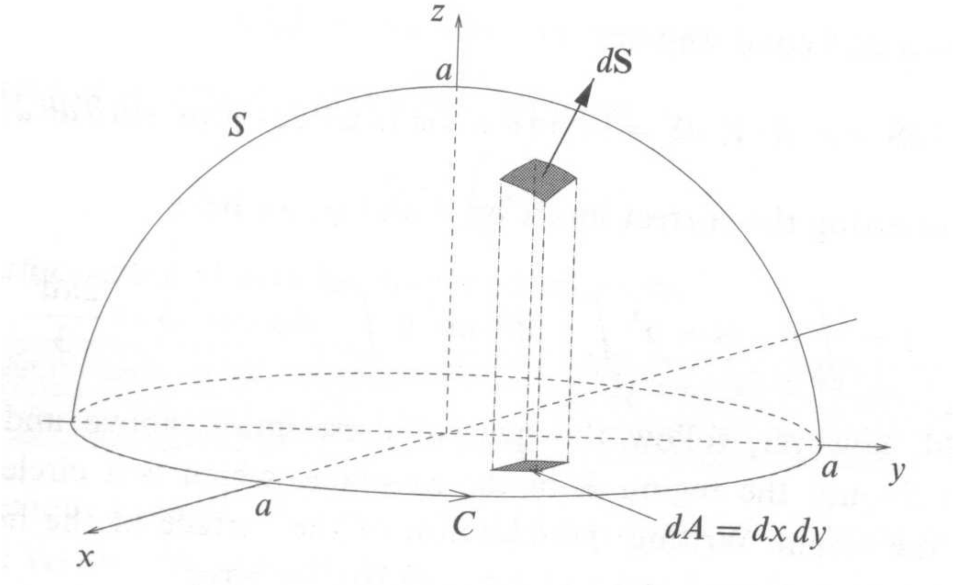

FIG. 6: The surface of the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$.

In this case dS may be easily expressed in spherical polar coordinates as $dS = a^2 \sin\theta \, d\theta \, d\phi$, and the unit normal to the surface at any point is simply $\hat{\mathbf{r}}$.

On the surface of the hemisphere, we have $x = a\sin\theta\cos\phi$ and so

$$\begin{aligned}\mathbf{a}\cdot d\mathbf{S} &= x(\mathbf{i}\cdot\hat{\mathbf{r}})\,dS \\ &= (a\sin\theta\cos\phi)(\sin\theta\cos\phi)(a^2\sin\theta\,d\theta\,d\phi)\end{aligned}$$

Therefore inserting the correct limits on θ and ϕ, we have

$$\begin{aligned}I &= \int_S \mathbf{a}\cdot d\mathbf{S} \\ &= a^3\int_0^{\pi/2} d\theta\,\sin^3\theta\int_0^{2\pi} d\phi\,\cos^2\phi \\ &= \frac{2\pi a^3}{3}\end{aligned}$$

We could, however, follow the general prescription above and project the hemisphere S onto the region R in the xy-plane, which is a circle of radius a centered at the origin. writing the equation of the surface of the hemisphere as $f(x, y) = x^2 + y^2 + z^2 - a^2 = 0$ and using Eq. (6), we have

$$\begin{aligned} I &= \int_S \mathbf{a} \cdot d\mathbf{S} = \int_S x(\mathbf{i} \cdot \hat{\mathbf{r}})\, dS \\ &= \int_R x(\mathbf{i} \cdot \hat{\mathbf{r}}) \frac{|\nabla f|\, dA}{\partial f / \partial z}. \end{aligned}$$

Now $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = 2\mathbf{r}$, so on the surface S we have $|\nabla f| = 2|\mathbf{r}| = 2a$. On S, we also have $\partial f / \partial z = 2z = 2\sqrt{a^2 - x^2 - y^2}$ and $\mathbf{i} \cdot \hat{\mathbf{r}} = x/a$.

Therefore, the integral becomes

$$I = \int\int_R \frac{x^2}{\sqrt{a^2 - x^2 - y^2}} dx\, dy.$$

Although this integral may be evaluated directly, it is quicker to transform to plane polar coordinates:

$$\begin{aligned} I &= \int\int_{R'} \frac{\rho^2 \cos^2\phi}{\sqrt{a^2 - \rho^2}} \rho\, d\rho\, d\phi \\ &= \int_0^{2\pi} \cos^2\phi\, d\phi \int_0^a \frac{\rho^3\, d\rho}{\sqrt{a^2 - \rho^2}} \end{aligned}$$

Making the substitution $\rho = a \sin u$, we finally obtain

$$\begin{aligned} I &= \int_0^{2\pi} \cos^2\phi\, d\phi \int_0^{\pi/2} a^3 \sin^3 u\, du \\ &= \frac{2\pi a^3}{3}. \end{aligned}$$

Vector areas of surfaces

The vector area of a surface S is defined as

$$\mathbf{S} = \int_S d\mathbf{S}$$

Example

Find the vector area of the surface of the hemisphere $x^2 + y^2 + z^2 = a^2$ with $z \geq 0$.

Answer

$d\mathbf{S} = a^2 \sin\theta \, d\theta \, d\phi \, \hat{\mathbf{r}}$ in spherical polar coordinates.

The vector area is

$$\mathbf{S} = \int\int_S a^2 \sin\theta \, \hat{\mathbf{r}} \, d\theta \, d\phi$$

Since $\hat{\mathbf{r}}$ varies over the surface S, it also must be integrated. On S we have

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k}$$

so

$$\begin{aligned}
\mathbf{S} &= \mathbf{i}\left(a^2 \int_0^{2\pi} \cos\phi \, d\phi \int_0^{\pi/2} \sin^2\theta \, d\theta\right) \\
&\quad +\mathbf{j}\left(a^2 \int_0^{2\pi} \sin\phi \, d\phi \int_0^{\pi/2} \sin^2\theta \, d\theta\right) \\
&\quad +\mathbf{k}\left(a^2 \int_0^{2\pi} d\phi \int_0^{\pi/2} \sin\theta\cos\theta \, d\theta\right) \\
&= \mathbf{0} + \mathbf{0} + \pi a^2 \mathbf{k} \\
&= \pi a^2 \mathbf{k}.
\end{aligned}$$

Projected area of the hemisphere onto the xy-plane.

Volume integrals

Examples:

$$\int_V \phi \, dV, \quad \int_V \mathbf{a} \, dV$$

Volumes of three-dimensional regions

The volume of a three-dimensional region V is simplpy $V = \int_V dV$. We shall now express it in terms of a surface integral over S.

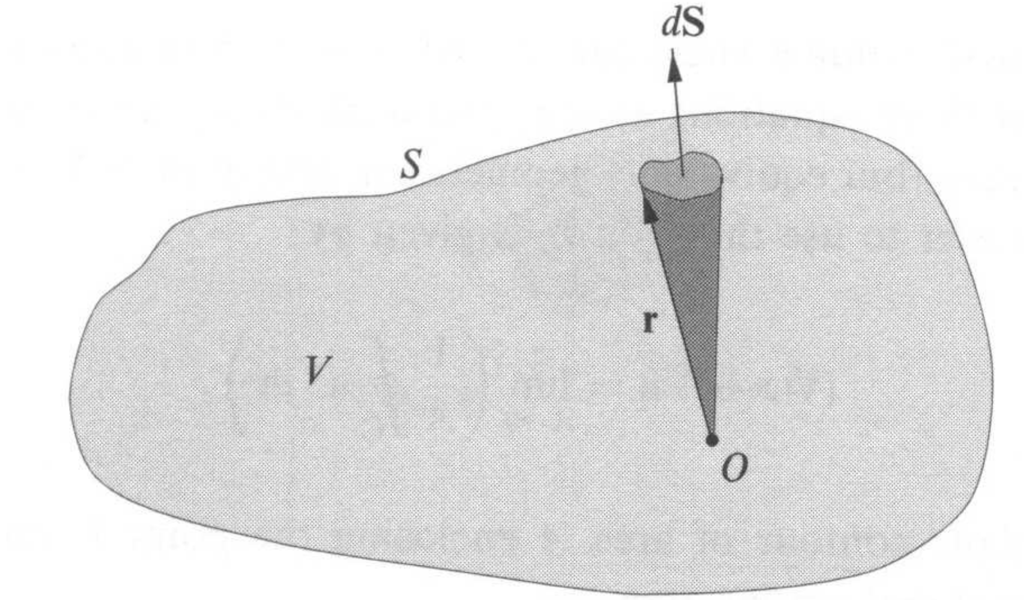

FIG. 7: A general volume V containing the origin and bounded by the closed surface S.

Let us suppose that the origin O is contained within the V. Then the volume of the small shaded cone is $dV = \frac{1}{3}\mathbf{r} \cdot d\mathbf{S}$. The total volume of the region is then given by

$$V = \frac{1}{3} \oint_S \mathbf{r} \cdot d\mathbf{S}$$

This expression is still valid even when O is not contained in V.

Example

Find the volume enclosed between a sphere of radius a centered on the origin, and a circular cone of half angle α with its vertex at the origin.

Answer

Now $d\mathbf{S} = a^2 \sin\theta \, d\theta \, d\phi \, \hat{\mathbf{r}}$. Taking the axis of the cone to lie along the z-axis (from which θ is measured) the required volume is given by

$$\begin{aligned} V &= \frac{1}{3} \oint_S \mathbf{r} \cdot d\mathbf{S} \\ &= \frac{1}{3} \int_0^{2\pi} d\phi \int_0^{\alpha} a^2 \sin\theta \, \mathbf{r} \cdot \hat{\mathbf{r}} \, d\theta \\ &= \frac{1}{3} \int_0^{2\pi} d\phi \int_0^{\alpha} a^3 \sin\theta \, d\theta \\ &= \frac{2\pi a^3}{3} (1 - \cos\alpha). \end{aligned}$$

Integral forms for grad, div and curl

At any point P, we have

$$\begin{aligned}
\nabla \phi &= \lim_{V \to 0} \left(\frac{1}{V} \oint_S \phi \, d\mathbf{S} \right) & (7) \\
\nabla \cdot \mathbf{a} &= \lim_{V \to 0} \left(\frac{1}{V} \oint_S \mathbf{a} \cdot d\mathbf{S} \right) & (8) \\
\nabla \times \mathbf{a} &= \lim_{V \to 0} \left(\frac{1}{V} \oint_S d\mathbf{S} \times \mathbf{a} \right) & (9) \\
(\nabla \times \mathbf{a}) \cdot \hat{\mathbf{n}} &= \lim_{A \to 0} \left(\frac{1}{A} \oint_C d\mathbf{a} \cdot d\mathbf{r} \right) & (10)
\end{aligned}$$

where V is a small volume enclosing P and S is its bounding surface. C is a plane contour area A enclosing the point P and $\hat{\mathbf{n}}$ is the unit normal to the enclosed planar area.

Divergence theorem and related theorems

Imagine a volume V, in which a vector field $\mathbf{a}$ is continuous and differentiable, to be divided up into a large number of small volumes V_i. Using Eq. (8), we have for each small volume,

$$(\nabla \cdot \mathbf{a})V_i \approx \oint_{S_i} \mathbf{a} \cdot d\mathbf{S},$$

where S_i is the surface of the small volume V_i. Summing over i, contributions from surface elements interior to S cancel, since each surface element appears in two terms with opposite signs. Only contributions from surface elements which are also parts of S survive. If each V_i is allowed to tend to zero, we obtain the divergence theorem

$$\int_V \nabla \cdot \mathbf{a}\, dV = \oint_S \mathbf{a} \cdot d\mathbf{S} \tag{11}$$

If we set $\mathbf{a} = \mathbf{r}$, we obtain

$$\int_V \nabla \cdot \mathbf{r}\, dV = \int_V 3\, dV = 3V = \oint_S \mathbf{r} \cdot d\mathbf{S}$$

Example

Evaluate the surface integral $I = \int_S \mathbf{a} \cdot d\mathbf{S}$, where $\mathbf{a} = (y - x)\mathbf{i} + x^2 z\mathbf{j} + (z + x^2)\mathbf{k}$, and S is the open surface of the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$.

Answer

Let us consider a closed surface $S' = S + S_1$, where S_1 is the circular area in the xy-plane given by $x^2 + y^2 \leq a^2$, $z = 0$; S' then encloses a hemispherical volume V. By the divergence theorem, we have

$$\int_V \nabla \cdot \mathbf{a}\, dV = \oint_{S'} \mathbf{a} \cdot d\mathbf{S} = \int_S \mathbf{a} \cdot d\mathbf{S} + \int_{S_1} \mathbf{a} \cdot d\mathbf{S}$$

Now $\nabla \cdot \mathbf{a} = -1 + 0 + 1 = 0$, so we can write

$$\int_S \mathbf{a} \cdot d\mathbf{S} = -\int_{S_1} \mathbf{a} \cdot d\mathbf{S}$$

The surface element on S_1 is $d\mathbf{S} = -\mathbf{k}\, dx\, dy$.

On S_1, we also have $\mathbf{a} = (y - x)\mathbf{i} + x^2\mathbf{k}$, so that

$$I = -\int_{S_1} \mathbf{a} \cdot d\mathbf{S} = \int\int_R x^2\,dx\,dy,$$

where R is the circular region in the xy-plane given by $x^2 + y^2 \leq a^2$. Transforming to plane polar coordinates, we have

$$\begin{aligned} I &= \int\int_{R'} \rho^2 \cos^2\phi\rho\,d\rho\,d\phi \\ &= \int_0^{2\pi} \cos^2\phi\,d\phi \int_0^a \rho^3\,d\rho \\ &= \frac{\pi a^4}{4}. \end{aligned}$$

Green's theorem

Consider two scalar functions ϕ and ψ that are continuous and differentiable in some volume V bounded by a surface S. Applying the divergence theorem to the vector field $\phi\nabla\psi$, we obtain

$$\begin{aligned}\oint_S \phi\nabla\psi \cdot d\mathbf{S} &= \int_V \nabla \cdot (\phi\nabla\psi)\, dV \\ &= \int_V \left[\phi\nabla^2\psi + (\nabla\phi)\cdot(\nabla\psi)\right]\, dV\end{aligned} \tag{12}$$

Reversing the roles of ϕ and ψ in Eq. (12) and subtracting the two equations gives

$$\oint_S (\phi\nabla\psi - \psi\nabla\phi)\cdot d\mathbf{S} = \int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, dV \tag{13}$$

Equation (12) is usually known as Green's first theorem and (13) as his second.

Other related integral theorems

If ϕ is a scalar field and $\mathbf{b}$ is a vector field and both satisfy the differentiability conditions in some volume V bounded by a closed surface S, then

$$\begin{aligned} \int_V \nabla\phi \, dV &= \oint_S \phi \, d\mathbf{S} \qquad (14) \\ \int_V \nabla\times\mathbf{b} \, dV &= \oint_S d\mathbf{S}\times\mathbf{b} \qquad (15) \end{aligned}$$

Proof of Eq. (14)

In Eq. (11), let $\mathbf{a} = \phi\mathbf{c}$, where $\mathbf{c}$ is a constant vector. We then have

$$\int_V \nabla\cdot(\phi\mathbf{c}) \, dV = \oint_S \phi\mathbf{c}\cdot d\mathbf{S}$$

Expanding out the integrand on the LHS, we have

$$\nabla\cdot(\phi\mathbf{c}) = \phi\nabla\cdot\mathbf{c} + \mathbf{c}\cdot\nabla\phi = \mathbf{c}\cdot\nabla\phi.$$

Also, $\phi\mathbf{c}\cdot d\mathbf{S} = \mathbf{c}\cdot\phi d\mathbf{S}$, so we obtain

$$\mathbf{c}\cdot\int_V \nabla\phi \, dV = \mathbf{c}\cdot\oint_S \phi \, d\mathbf{S}.$$

Since $\mathbf{c}$ is arbitrary, we obtain the stated result.

Example

For a compressible fluid with time-varying position-dependent density $\rho(\mathbf{r}, t)$ and velocity field $v(\mathbf{r}, t)$, in which fluid is neither being created nor destroyed, show that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Answer

For an arbitrary volume in the fluid, conservation of mass says that the rate of increase or decrease of the mass M in the volume must equal the net rate at which fluid is entering or leaving the volume, i.e.

$$\frac{dM}{dt} = -\oint_S \rho \mathbf{v} \cdot d\mathbf{S},$$

where S is the surface bounding V. But the mass of fluid in V is $M = \int_V \rho \, dV$, so we have

$$\frac{d}{dt} \int_V \rho \, dV + \oint_S \rho \mathbf{v} \cdot d\mathbf{S} = 0$$

Using the divergence theorem, we have

$$\int_V \frac{\partial\rho}{\partial t} dV + \int_V \nabla\cdot(\rho\mathbf{v})\, dV = \int_V \left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})\right] dV = 0$$

Since the volume V is arbitrary, the integrand must be identically zero, so

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

Stokes' theorem and related theorems

Following the same lines as for the derivation of the divergence theorem, we can divide the surface S into many small areas S_i with boundaries C_i and with unit normals $\hat{\mathbf{n}}_i$. Using Eq. (10), we have for each small area

$$(\nabla \times \mathbf{a}) \cdot \hat{\mathbf{n}}_i S_i \approx \oint_{C_i} \mathbf{a} \cdot d\mathbf{r}$$

Summing over i we find that on the RHS all parts of all interior boundaries that are not part of C are included twice, being traversed in opposite directions on each occasion and thus contributing nothing. Only contributions from line elements that are also parts of C survive. If each S_i is allowed to tend to zero, we obtain Stokes' theorem,

$$\int_S (\nabla \times \mathbf{a}) \cdot d\mathbf{S} = \oint_C \mathbf{a} \cdot d\mathbf{S} \qquad (16)$$

Example

Given the vector field $\mathbf{a} = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$, verify Stokes' theorem for the hemispherical surface $x^2 + y^2 + z^2 = a^2$, $z \geq 0$.

Answer

Let us evaluate the surface integral

$$\int_S (\nabla \times \mathbf{a}) \cdot d\mathbf{S},$$

over the hemisphere. Since $\nabla \times \mathbf{a} = -2\mathbf{k}$ and the surface element is $d\mathbf{S} = a^2 \sin\theta \, d\theta \, d\phi \, \hat{\mathbf{r}}$, we have

$$\begin{aligned}
\int_S (\nabla \times \mathbf{a}) \cdot d\mathbf{S} &= \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta (-2a^2 \sin\theta) \hat{\mathbf{r}} \cdot \mathbf{k} \\
&= -2a^2 \int_0^{2\pi} d\phi \int_0^{\pi/2} \sin\theta \left(\frac{z}{a}\right) d\theta \\
&= -2a^2 \int_0^{2\pi} d\phi \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta \\
&= -2\pi a^2.
\end{aligned}$$

The line integral around the perimeter curve C (a circle $x^2 + y^2 = a^2$ in the xy-plane) is given by

$$\begin{aligned}\oint_C \mathbf{a} \cdot d\mathbf{r} &= \oint_C (y\mathbf{i} - x\mathbf{j} + z\mathbf{k}) \cdot (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}) \\ &= \oint_C (y\,dx - x\,dy)\end{aligned}$$

Using plane polar coordinates, on C we have $x = a\cos\phi$, $y = a\sin\phi$ so that $dx = -a\sin\phi\,d\phi$, $dy = a\cos\phi\,d\phi$, and the line integral becomes

$$\begin{aligned}\oint_C (y\,dx - x\,dy) &= -a^2 \int_0^{2\pi} (\cos^2\phi + \sin^2\phi)\,d\phi \\ &= -a^2 \int_0^{2\pi} d\phi = -2\pi a^2\end{aligned}$$

Since the surface and line integrals have the same value, we have verified Stokes' theorem.

Related integral theorems

$$\int_S d\mathbf{S} \times \nabla\phi = \oint_C \phi \, d\mathbf{r} \tag{17}$$

$$\int_S (d\mathbf{S} \times \nabla) \times \mathbf{b} = \oint_C d\mathbf{r} \times \mathbf{b} \tag{18}$$

Proof of Eq. (17)

In Stokes' theorem, Eq. (16), let $\mathbf{a} = \phi\mathbf{c}$, where $\mathbf{c}$ is a constant vector. We then have

$$\int_S [\nabla \times (\phi\mathbf{c})] \cdot d\mathbf{S} = \oint_C \phi\mathbf{c} \cdot d\mathbf{r} \tag{19}$$

Expanding out the integrand on the LHS, we have

$$\nabla \times (\phi\mathbf{c}) = \nabla\phi \times \mathbf{c} + \phi\nabla \times \mathbf{c} = \nabla\phi \times \mathbf{c}.$$

so that

$$[\nabla \times (\phi\mathbf{c})] \cdot d\mathbf{S} = (\nabla\phi \times \mathbf{c}) \cdot d\mathbf{S} = \mathbf{c} \cdot (d\mathbf{S} \times \nabla\phi).$$

Substituting this into Eq. (19), and taking $\mathbf{c}$ out of both integrals, we find

$$\mathbf{c}\cdot\int_S d\mathbf{S}\times\nabla\phi=\mathbf{c}\cdot\oint_C \phi\, d\mathbf{r}$$

Since $\mathbf{c}$ is an arbitrary constant vector, we therefore obtain the stated result.