Integral Transforms

Fourier transforms

The Fourier transform provides a representation of functions defined over an infinite interval, and having no particular periodicity, in terms of superposition of sinusoidal functions.

A function of period T may be represented as a complex Fourier series,

$$f(t) = \sum_{r=-\infty}^{\infty} c_r e^{2\pi i r t/T} = \sum_{r=-\infty}^{\infty} c_r e^{i\omega_r t} \qquad (1)$$

where $\omega_r = 2\pi r/T$. As the period T tends to infinity, the 'frequency quantum' $\Delta\omega = 2\pi/T$ becomes vanishingly small and the spectrum of allowed frequencies ω_r becomes a continuum. Thus the infinite sum of terms in the Fourier series becomes an integral, and the coefficients c_r become functions of the continuous variable ω.

We recall that the coefficients c_r in Eq. (1) are given by

$$\begin{aligned} c_r &= \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-2\pi i r t/T}\,dt \\ &= \frac{\Delta\omega}{2\pi}\int_{-T/2}^{T/2} f(t)e^{-i\omega_r t}\,dt. \end{aligned} \tag{2}$$

Substituting from Eq. (2) and into (1) gives

$$f(t) = \sum_{r=-\infty}^{\infty} \frac{\Delta\omega}{2\pi}\int_{-T/2}^{T/2} f(u)e^{-i\omega_r u}\,du e^{i\omega_r t}. \tag{3}$$

At this stage, ω_r is still a discrete function of r equal to $2\pi r/T$.

The solid points in the figure below are a plot of $c_r e^{i\omega_r t}$ as a function of r and $(2\pi/T)c_r e^{i\omega_r t}$ gives the area of the rth (broken line) rectangle. If T tends to ∞, $\Delta\omega(= 2\pi/T)$ consequently becomes infinitesimal, the width of the rectangles tends to zero.

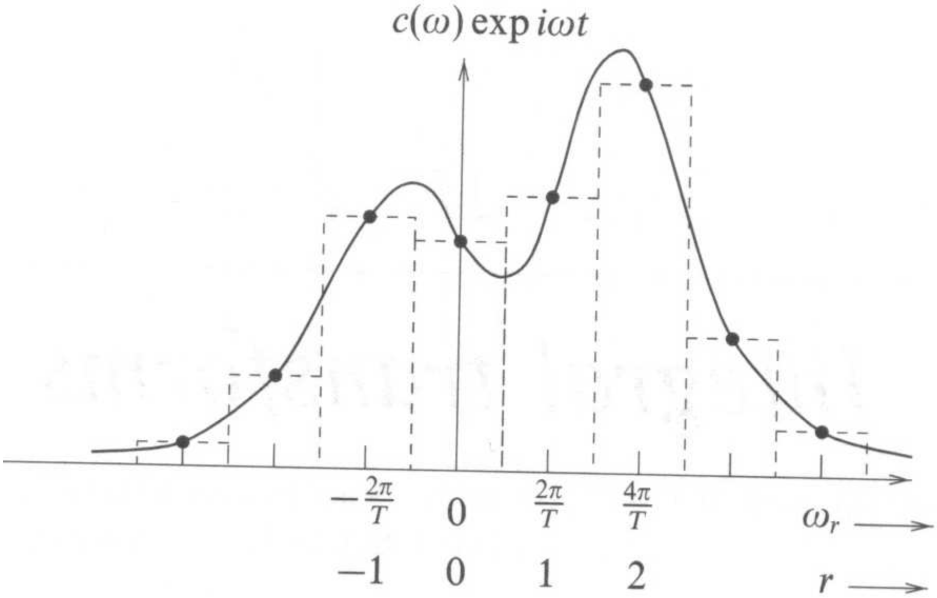

FIG. 1: The relationship between the Fourier terms for a function of period T and the Fourier integral (the area below the solid line) of the function.

Also,

$$\sum_{r=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} g(\omega_r) e^{i\omega_r t} \to \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\omega) e^{i\omega t}\, d\omega$$

In this particular case,

$$g(\omega_r) = \int_{-T/2}^{T/2} f(u) e^{-i\omega_r u}\, du,$$

and Eq. (3) becomes

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega t} \int_{-\infty}^{\infty} du\, f(u) e^{-i\omega u}. \qquad (4)$$

This result is known as Fourier's inversion theorem.

From it, we may define Fourier transform of $f(t)$ by

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt, \qquad (5)$$

and its inverse by

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{i\omega t}\, d\omega. \qquad (6)$$

Example

Find the Fourier transform of the exponential decay function $f(t) = 0$ for $t < 0$ and $f(t) = Ae^{-\lambda t}$ for $t \geq 0$ $(\lambda > 0)$.

Answer

Using Eq. (5) and separating the integrals into two parts,

$$\begin{aligned}
\tilde{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} (0) e^{-i\omega t} dt \\
&\quad + \frac{A}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\lambda t} e^{-i\omega t} dt \\
&= 0 + \frac{A}{\sqrt{2\pi}} \left[-\frac{e^{-(\lambda + i\omega)t}}{\lambda + i\omega} \right]_{0}^{\infty} \\
&= \frac{A}{\sqrt{2\pi}(\lambda + i\omega)}.
\end{aligned}$$

Dirac δ-function

The δ-function can be visualized as a very sharp narrow pulse (in space, time, density, etc) which produces an integrated effect of definite magnitude.

The Dirac δ-function has the property that

$$\delta(t) = 0 \quad \text{for } t \neq 0, \tag{7}$$

but its fundamental defining property is

$$\int f(t)\delta(t-a)\,dt = f(a) \tag{8}$$

provided the range of integration includes the point $t = a$; otherwise the integral equals zero. This leads to two further results:

$$\int_{-a}^{b} \delta(t)\,dt = 1 \quad \text{for all } a, b > 0 \tag{9}$$

and

$$\int \delta(t-a)\,dt = 1 \tag{10}$$

provided the range of integration includes $t = a$.

Eq. (8) can be used to derive further useful properties of the Dirac δ-function:

$$\delta(t) = \delta(-t) \tag{11}$$

$$\delta(at) = \frac{1}{|a|}\delta(t) \tag{12}$$

$$t\delta(t) = 0. \tag{13}$$

Example

Prove that $\delta(bt) = \delta(t)/|b|$.

Answer

Let us consider the case where $b > 0$. It follows that

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)\delta(bt)\,dt &= \int_{-\infty}^{\infty} f\left(\frac{t'}{b}\right)\delta(t')\frac{dt'}{b} \\ &= \frac{1}{b}f(0) = \frac{1}{b}\int_{-\infty}^{\infty} f(t)\delta(t)\,dt,\end{aligned}$$

where we have made the substitution $t' = bt$. But $f(t)$ is arbitrary and therefore $\delta(bt) = \delta(t)/b = \delta(t)/|b|$ for $b > 0$.

Now consider the case where $b = -c < 0$. It follows that

$$\begin{aligned}
\int_{-\infty}^{\infty} f(t)\delta(bt)\,dt &= \int_{\infty}^{-\infty} f\left(\frac{t'}{-c}\right)\delta(t')\left(\frac{dt'}{-c}\right) \\
&= \int_{-\infty}^{\infty} \frac{1}{c} f\left(\frac{t'}{-c}\right)\delta(t')\,dt' \\
&= \frac{1}{c} f(0) = \frac{1}{|b|} f(0) \\
&= \frac{1}{|b|}\int_{-\infty}^{\infty} f(t)\delta(t)\,dt
\end{aligned}$$

where we have made the substitution $t' = bt = -ct$. But $f(t)$ is arbitrary and so

$$\delta(bt) = \frac{1}{|b|}\delta(t),$$

for all b.

Relation of the δ-function to Fourier transforms

Referring to Eq. (4), we have

$$
\begin{aligned}
f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega t}\int_{-\infty}^{\infty} du\, f(u)e^{-i\omega u} \\
&= \int_{-\infty}^{\infty} du\, f(u)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-u)}\, d\omega\right\}
\end{aligned}
$$

Comparison of this with Eq. (8) shows that we may write the δ-function as

$$
\delta(t-u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-u)}\, d\omega. \qquad (14)
$$

The Fourier transform of a δ-function is

$$
\tilde{\delta}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \delta(t)e^{-i\omega t}\, dt = \frac{1}{\sqrt{2\pi}}. \qquad (15)
$$

Properties of Fourier transform

We denote the Fourier transform of $f(t)$ by $\tilde{f}(\omega)$ or $\mathcal{F}$.

1. Differentiation:

$$\mathcal{F}\left[f'(t)\right] = i\omega\tilde{f}(\omega) \tag{16}$$

   This may be extended to higher derivatives, so that

$$\mathcal{F}\left[f''(t)\right] = i\omega\mathcal{F}[f'(t)] = -\omega^2\tilde{f}(\omega),$$

   and so on.

2. Integration:

$$\mathcal{F}\left[\int^t f(s)\,ds\right] = \frac{1}{i\omega}\tilde{f}(\omega) + 2\pi c\delta(\omega), \tag{17}$$

   where the term $2\pi c\delta(\omega)$ represents the Fourier transform of the constant of integration associated with the indefinite integral.

3. Scaling:

$$\mathcal{F}[f(at)] = \frac{1}{a}\tilde{f}\left(\frac{\omega}{a}\right). \tag{18}$$

4. Translation:

$$\mathcal{F}[f(t+a)] = e^{ia\omega}\tilde{f}(\omega). \qquad (19)$$

5. Exponential multiplication:

$$\mathcal{F}\left[e^{\alpha t}f(t)\right] = \tilde{f}(\omega + i\alpha), \qquad (20)$$

where α may be real, imaginary or complex.

Example

Prove relation $\mathcal{F}\left[f'(t)\right] = i\omega\tilde{f}(\omega)$.

Answer

Calculating the Fourier transform of $f'(t)$ directly, we obtain

$$\begin{aligned}\mathcal{F}\left[f'(t)\right] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(t)e^{-i\omega t}\,dt \\ &= \frac{1}{\sqrt{2\pi}}\left[e^{-i\omega t}f(t)\right]_{-\infty}^{\infty} \\ &\quad +\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} i\omega e^{-i\omega t}f(t)\,dt \\ &= i\omega\tilde{f}(\omega),\end{aligned}$$

if $f(t)\to 0$ at $t=\pm\infty$ (as it must since $\int_{-\infty}^{\infty}|f(t)|\,dt$ is finite).

Odd and even functions

Let us consider an odd function $f(t) = -f(-t)$, whose Fourier transform is given by

$$\begin{aligned}\tilde{f}(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)(\cos\omega t - i\sin\omega t)\,dt \\ &= \frac{-2i}{\sqrt{2\pi}}\int_{0}^{\infty} f(t)\sin\omega\,dt,\end{aligned}$$

since $f(t)$ and $\sin\omega t$ are odd, whereas $\cos\omega t$ is even.

We note that $\tilde{f}(-\omega) = -\tilde{f}(\omega)$, i.e. $\tilde{f}(\omega)$ is an odd function of ω.

Hence

$$
\begin{aligned}
f(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{i\omega t}\, d\omega \\
&= \frac{2i}{\sqrt{2\pi}} \int_{0}^{\infty} \tilde{f}(\omega) \sin \omega t\, d\omega \\
&= \frac{2}{\pi} \int_{0}^{\infty} d\omega\, \sin \omega t \left\{ \int_{0}^{\infty} f(u) \sin \omega u\, du \right\}.
\end{aligned}
$$

Thus we may define the Fourier sine transform pair for odd functions:

$$
\tilde{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t) \sin \omega t\, dt, \tag{21}
$$

$$
f(t) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \tilde{f}_s(\omega) \sin \omega t\, d\omega. \tag{22}
$$

Convolution and deconvolution

The convolution of the functions f and g is defined as

$$h(z) = \int_{-\infty}^{\infty} f(x)g(z-x)\,dx \tag{23}$$

and is often written as $f * g$.

The convolution is commutative $\left(f * g = g * f\right)$, associative and distributive.

Example

Find the convolution of the function $f(x) = \delta(x+a) + \delta(x-a)$ with the function $g(y)$ plotted in the Figure below.

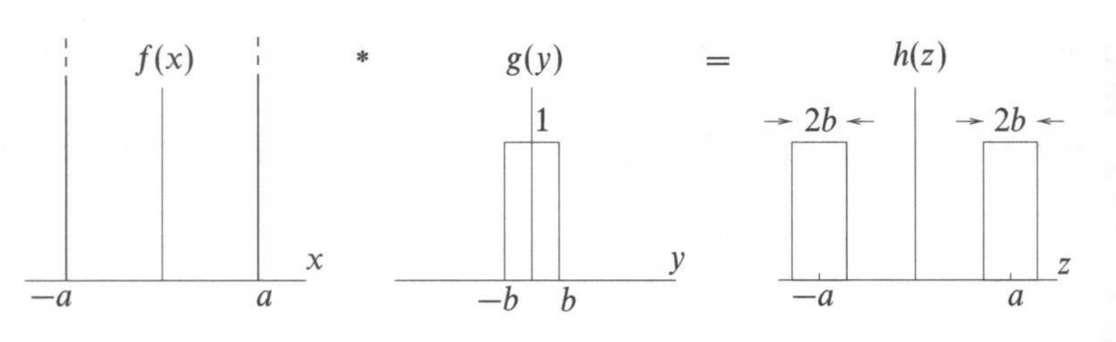

FIG. 2: The convolution of two functions $f(x)$ and $g(y)$.

Answer

Using the convolution integral Eq. (23),

$$\begin{aligned} h(z) &= \int_{-\infty}^{\infty} f(x) g(z-x)\, dx \\ &= \int_{-\infty}^{\infty} [\delta(x+a) + \delta(x-a)] g(z-x)\, dx \\ &= g(z+a) + g(z-a). \end{aligned}$$

Let us now consider the Fourier transform of the convolution [Eq. (23)], which is given by

$$\begin{aligned}\tilde{h}(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dz\, e^{-ikz}\left\{\int_{-\infty}^{\infty} f(x)g(z-x)\,dx\right\}\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx f(x)\left\{\int_{-\infty}^{\infty} g(z-x)e^{-ikz}\,dz\right\}\end{aligned}$$

If we let $u = z - x$ in the second integral we have

$$\begin{aligned}\tilde{h}(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx f(x)\left\{\int_{-\infty}^{\infty} g(u)e^{-ik(u+x)}\,du\right\}\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx\int_{-\infty}^{\infty} g(u)e^{-iku}\,du\\ &= \frac{1}{\sqrt{2\pi}}\times\sqrt{2\pi}\tilde{f}(k)\times\sqrt{2\pi}\tilde{g}(k)\\ &= \sqrt{2\pi}\tilde{f}(k)\tilde{g}(k). \qquad (24)\end{aligned}$$

Hence the Fourier transform of a convolution is equal to the product of the separate Fourier transforms multiplied by $\sqrt{2\pi}$; this is called the convolution theorem.

The converse is also true, namely, that the Fourier transform of the product $f(x)g(x)$ is given by

$$\mathcal{F}[f(x)g(x)] = \frac{1}{\sqrt{2\pi}}\tilde{f}(k) * \tilde{g}(k). \qquad (25)$$

Fourier transform in higher dimensions

Fourier transform of $f(x,y,z)$ is

$$\tilde{f}(k_x,k_y,k_z) = \frac{1}{(2\pi)^{3/2}}\int\int\int f(x,y,z)e^{-ik_x x}e^{-ik_y y}e^{-ik_z z}dxdydz \quad (26)$$

and its inverse by

$$f(x,y,z) = \frac{1}{(2\pi)^{3/2}}\int\int\int \tilde{f}(k_x,k_y,k_z)e^{ik_x x}e^{ik_y y}e^{ik_z z}dk_xdk_ydk_z \quad (27)$$

Denoting the vector with components k_x, k_y, k_z by $\mathbf{k}$ and that with components x, y, z by $\mathbf{r}$, we can write the Fourier transform pair Eqs. (26), (27) as

$$\tilde{f}(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}}\int f(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}\,d^3\mathbf{r} \quad (28)$$

$$f(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}}\int \tilde{f}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}\,d^3\mathbf{k} \quad (29)$$

We may deduce that the three-dimensional Dirac δ-function can be written as

$$\delta(\mathbf{r}) = \frac{1}{(2\pi)^3}\int e^{i\mathbf{k}\cdot\mathbf{r}}\,d^3\mathbf{k}. \quad (30)$$