Physics 125a – Problem Set 4 – Due Oct 29, 2007

Version 2 – Oct 22, 2007

This problem set focuses entirely on the infinite-dimensional generalization of inner-product spaces, Shankar Section 1.10 and Lecture Notes Section 3.9.
v. 2: Clarifications/hints on Problems 1, 3, and 4 added.

1. Shankar 1.10.2 + extra: Show that

$$\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{\left| \left.\frac{df}{dx}\right|_{x=x_i} \right|} \tag{1}$$

   where $\{x_i\}$ are the zeros of the function $f(x)$, $f(x_i) = 0$. You also should assume $\left.\frac{df}{dx}\right|_{x=x_i} \neq 0$ and is finite for all x_i so the above formula is well-defined. Hint: where does $\delta(f(x))$ blow up? Expand $f(x)$ near such points in a Taylor series, keeping the first term, and do a change of integration variable. You do *not* need to prove the first item below in order to prove the above theorem.

   Using the above, show the following:

   - $\delta(ax) = \frac{1}{|a|}\,\delta(x)$
   - $\delta(x^2 - a^2) = \frac{1}{2|a|}\left[\delta(x - a) + \delta(x + a)\right]$

   (Use Equation 1 to do these even if you are unable to prove Equation 1.)

2. Use a change of integration variable to show that $\delta(\sqrt{x}) = 0$ and use your result to evaluate $\delta(\sqrt{x^2 - a^2})$. (This is not just a matter of applying Equation 1 again because $\left.\frac{df}{dx}\right|_{x=0}$ is infinite.)

3. Shankar 1.10.3: Use integration by parts over an infinitesimal interval around $x = x'$ to show

$$\frac{d}{dx}\,\theta(x - x') = \delta(x - x') \quad \text{where} \quad \theta(x - x') = \begin{cases} 0 & x < x' \\ 1 & x \geq x' \end{cases}$$

4. We have proven the following relations for derivatives of the δ function:

$$\left[\frac{d}{dx}\,\delta(x - x')\right] = \delta(x - x')\frac{d}{dx'} \qquad \left[\frac{d}{dx'}\,\delta(x - x')\right] = -\delta(x - x')\frac{d}{dx'}$$

   Prove these more rigorously by using the Gaussian approximation to the δ function and taking the appropriate limits. Note that you will have to do an integration by parts in order to make sense of the factor $\frac{d}{dx'}$ acting to the right.

5. In class, we considered the space of complex functions with complex coefficients on the real line. Our kets $|f\rangle$ were defined via their $\{|x\rangle\}$ basis representation, $\langle x|f\rangle = f(x)$. We then showed how they could be rewritten in terms of the $\{|k\rangle\}$ basis by Fourier transforming:

$$\widetilde{f}(k) = \langle k|f\rangle = \int_{-\infty}^{\infty} dx \langle k|x\rangle\langle x|f\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-ikx} f(x)$$

We defined the inner product $\langle f|g\rangle$ explicitly in terms of the $\{|x\rangle\}$ representation:

$$\langle f|g\rangle = \int_{-\infty}^{\infty} dx\, f^*(x)\, g(x)$$

Rewrite the inner product formula in terms of the $\{|k\rangle\}$ representation of $|f\rangle$ and $|g\rangle$; *i.e.*, in terms of $\widetilde{f}(k)$ and $\widetilde{g}(k)$. You will need to use completeness and orthonormality.