Physics 125a – Problem Set 3 – Due Oct 22, 2007

Version 2 – Oct 18, 2007

This problem set provides some more practice on eigenvalue problems and also covers functions of and calculus with operators; Shankar Sections 1.8-1.9 and Lecture Notes Section 3.6-3.8.

v. 2: A few typos/omissions:

Problem 1: type, "two operators", not "three operators"

Problem 2: E_0 and W should be real, mistakenly used λ for the eigenvalue. Forgot the "fractional" in "fractional quantum hall effect."

Problem 4b: prove only via eigenbasis representation

1. Consider a 2-dimensional inner product space with orthonormal basis kets $|1\,\rangle$ and $|2\,\rangle$. Suppose there are Hermitian (but possibly complex) operators P and R acting on the space with known matrix elements

$$\langle 1\,|\,P\,|1\,\rangle = \frac{1}{2} \qquad \langle 1\,|\,P^2\,|1\,\rangle = \frac{1}{4} \tag{1}$$

$$\langle 1\,|\,R\,|1\,\rangle = 1 \qquad \langle 1\,|\,R^2\,|1\,\rangle = \frac{5}{4} \qquad \langle 1\,|\,R^3\,|1\,\rangle = \frac{7}{4} \tag{2}$$

Write down matrix representations for the two operators in this basis in terms of as few free parameters as possible. Find the eigenvalues of the two operators in terms of these free parameters.

2. Consider a n-dimensional inner product space with orthonormal basis $\{|j\,\rangle\}$, $j = 1, \ldots, n$. Consider the operators

$$\Gamma = \sum_{j=1}^{n} E_0 |j\,\rangle\langle j\,| \qquad \Delta = \sum_{j=1}^{n} W\Big[|j\,\rangle\langle j+1\,| + |j+1\,\rangle\langle j\,|\Big] \qquad \Omega = \Gamma + \Delta \tag{3}$$

where E_0 and W are **real numbers**. Assume "periodicity" in the labeling so that $|j{+}n\,\rangle = |j\,\rangle$ (this lets us write the operators in compact form.) Answer the following with appropriate justification:

(a) Are Γ, Δ, and Ω Hermitian? How is the Hermiticity of Δ dependent on the definition of Δ as the sum of two terms instead of just using one or the other?

(b) Calculate $[\Gamma, \Delta]$. There is an easy way to do this and a hard way, and it requires a bit of thought to show that the result from the hard way is the same as that from the easy way.

(c) Find the eigenvalues and eigenvectors of Ω. You will find it difficult to do this by direct calculation of the determinant. Rather, given the repetitive structure of Ω (Ω is known as a *circulant* or *Toeplitz* matrix), assume that the eigenvectors $\{v_l\}$ ($l = 1, \ldots, n$) are such that the kth component of the lth eigenvector is

$$\langle k\,|v_l\,\rangle \propto e^{ik\theta_l} \quad \text{with} \quad \theta_l = \frac{2\pi}{n}\,l, \quad l = 1, \ldots, n \tag{4}$$

Consider the kth row of the characteristic equation; *i.e.*

$$[(\Omega - \omega\, I)\, |v_l\,\rangle]_k = 0 \tag{5}$$

and obtain a relationship between ω and θ_l in terms of E_0 and W. Sketch a graph of the eigenvalue ω_l as a function of θ_l.

Physically, Ω could be the Hamiltonian (energy operator) for a system consisting of electrons on a set of isolated sites (the $|j\,\rangle$ states) but with some sort of coupling between the sites allowing movement of electrons from one site to the next (e.g., via thermal excitation). The resulting eigenvectors are not localized on a single site, but spread out among the sites with a site-dependent phase shift, and obtain a range of energies instead of just the energy E_0 of an electron localized on a single site. Such a Hamiltonian is a good first approximation for the valence electrons in many conducting and semiconducting crystalline solids.

This technique of guessing a form for the eigenvectors and checking that it is correct – making an *Ansatz* – is frequently used in physics when a direct solution is computationally intractable. Laughlin's Nobel Prize for explaining the fractional quantum Hall effect was basically a matter of doing this (in a much harder problem!). It is good to learn how to do this because it develops your ability to see through the algebra of a problem to what the characteristics of the solution must be.

3. Shankar 1.9.1: We know that for numbers x,

$$(1 - x)^{-1} = \sum_{n=0}^{\infty} x^n \tag{6}$$

if $|x| < 1$. By going to the eigenbasis of the Hermitian operator Ω, under what conditions does it hold that

$$(1 - \Omega)^{-1} = \sum_{n=0}^{\infty} \Omega^n \tag{7}$$

4. Shankar 1.9.2/1.9.3

   (a) If H is Hermitian, show that $U = e^{iH}$ is unitary. Prove this in two ways: first, by consideration in H's eigenbasis; and second, by power series expansion.

   (b) Using the above, show via consideration in H's eigenbasis that $\det U = e^{i\,\mathrm{Tr}\,H}$. Why is it good enough to show this in the eigenbasis?

5. For Ω a Hermitian operator and λ a (possibly complex) number, write down the power series expansions for $\sin(\lambda\Omega)$ and $\cos(\lambda\Omega)$. Show that $\frac{d}{d\lambda}\sin(\lambda\Omega) = \Omega\cos(\lambda\Omega)$ via the power series expansions.