Physics 125b – Midterm Exam – Due Feb 12, 2008

Instructions

Material: All lectures through Jan 25, lecture notes Sections 11-13. This corresponds roughly to Shankar Chapters 12 and 16, but if material from Shankar was not covered by or pointed to in the lecture notes or homework, it will not be needed here. Review the material ahead of time, consult me, the TAs, your fellow students, or other texts if there is material you are having trouble with.

Logistics: The exam consists of this page plus 2 pages of exam questions, a total of 5 questions. Do not look at the exam until you are ready to start it. Please use a blue book if possible (makes grading easier), but there will be no penalty if you don't have one.

Time: 4 hrs, fixed time. You may take as many breaks as you like, but they may add up to no more than 30 minutes (2 x 15 minutes, 3 x 10 minutes, etc.).

Reference policy: Shankar, official class lecture notes, problem sets and solutions, your own lecture notes or other notes you have taken to help yourself understand the material. No other textbooks, no web searches, no interaction with your fellow students. You may use a computer to write up your exam, but calculators and symbolic manipulation programs are neither needed or allowed. If you write up with a computer, it must be done within the 4-hour exam period; no additional time is allowed for transcription or proofreading. No dispensations will be given for technical difficulties. You may quote without proof any results given in the lecture notes, problem sets, or in Shankar.

Due date: Tuesday, Feb 12, 4 pm, my office (311 Downs). 4 pm means 4 pm. Late exams will require extenuating circumstances; otherwise, no credit will be given.

Grading: Each problem is 10 points for a total of 50 points. The exam is 1/3 of the class grade.

Suggestions on taking the exam:

- Go through and figure out roughly how to do each problem first; make sure you've got the concept straight before you start writing.
- Don't fixate on a particular problem. They are not all of equal difficulty. Come back to ones you are having difficulty with.
- Don't get buried in algebra. Get each problem to the point where you think you will get most of the points, then come back and worry about the algebra.

1. Consider the potential experienced by a ball bouncing against the ground under the influence of gravity:

$$V = \begin{cases} m\,g\,z & z > 0 \\ \infty & z \le 0 \end{cases} \tag{1}$$

Find the energy levels within the WKB approximation. (You do **not** need to find the wavefunctions!) Note: because the potential step is infinite, the WKB approximation breaks down miserably there. But you can relate the solutions of this problem to the solutions of the problem $V = m\,g\,|z|$, where there is no breakdown of WKB, and use the energy quantization condition of that problem here, taking into account the additional condition imposed by the wall on the wavefunction.

2. Recall our discussion of variational estimates of excited state energies in the lecture notes. There, we said that, if ϕ_0 and ϕ_1 are the true ground and first excited state wavefunctions with true energies E_0 and E_1, ψ_0 is a variational estimate for ϕ_0, and ψ_1 is a variational estimate for ϕ_1 that satisfies $\langle \psi_1 \,|\psi_0 \,\rangle = 0$, then it holds that

$$E[\psi_1] \ge E_1 - \delta_0\,(E_1 - E_0) \qquad \text{where} \qquad \delta_0 = 1 - |\langle \psi_0 \,|\phi_0 \,\rangle|^2 \tag{2}$$

Prove the above statement.

3. Let $\widehat{n}$ be a unit vector in a direction specified by the polar angles (θ, ϕ). Then, we have

$$L_n = \widehat{n} \cdot \vec{L} = \sin\theta\,\cos\,\phi\,L_x + \sin\theta\,\sin\,\phi\,L_y + \cos\theta\,L_z \tag{3}$$

$$= \frac{1}{2}\,\sin\theta\left(e^{-i\,\phi}L_+ + e^{i\,\phi}L_-\right) + \cos\theta\,L_z \tag{4}$$

If the system is in a simultaneous eigenstate of $\vec{L}^2$ and L_z with eigenvalues $l\,(l+1)\,\hbar^2$ and $m\,\hbar$,

(a) Assume $\theta \neq 0$ and $\theta \neq \pi$. Suppose one makes a measurement of L_n. What measurement outcomes are possible? That is, of the $2l + 1$ values that are available $(m = l, l - 1, \ldots, -l + 1, -l)$, are there any values that can never be measurement outcomes for all (θ, ϕ)? Note: there may be some directions (θ, ϕ) for which specific values of m do not arise due to particular cancellations; we are not asking you to find these, we are asking a more generic question. Hint: for angular momentum l, how many unique powers of $\hat{a} \cdot \vec{L}$ are there, where $\hat{a}$ is an arbitrary rotation vector?

(b) What are the expectation values of L_n and L_n^2?

4. Let's consider some of the implications of assuming that orbital angular momentum may take on half-integral values. Recall that one way to derive the spherical harmonics was to first use the requirement

$$\langle \theta, \phi \,|L_{\pm}|l, \pm l\,\rangle = 0 \tag{5}$$

to obtain a differential equation for $Y_l^{\pm l}$. (You can look this up in the notes.) Let's try to use this for $l = 1/2$.

   (a) What are the $Y_{1/2}^{\pm 1/2}$ that result (including normalization)?

   (b) What do you get if you act on $Y_{1/2}^{+1/2}$ with L_-? Is this consistent with the above result?

   (c) Show that the $Y_{1/2}^{\pm 1/2}$ derived above are inconsistent with the statement $\Pi^2 = I$ where Π is the parity operator.

   In the solutions, we will discuss the whether the above results are problematic or not. Feel free to think about it yourself...

5. Spherically symmetric problems:

   (a) Find the energy levels of a particle in a spherical box of radius r_0 for $l = 0$. A spherically symmetric box has the potential

$$V(r) = \begin{cases} 0 & 0 < r < r_0 \\ \infty & r \geq r_0 \end{cases} \tag{6}$$

   (b) A nonrelativistic particle of mass μ moves in a three-dimensional spherically symmetric potential $V(r)$ that vanishes for $r \to \infty$. We are given that the wavefunction of an eigenstate of the corresponding Hamiltonian is

$$\psi(r, \theta, \phi) = C\, r^{\sqrt{3}}\, e^{-\alpha\, r} \cos\theta \tag{7}$$

   where C and α are constants. (The $r^{\sqrt{3}}$ is not a mistake!) The state must therefore be an eigenstate of H, L^2, and L_z. What eigenvalue is obtained when L^2 acts on this state? L_z? What is the energy eigenvalue? What is $V(r)$? Hint: think about the $r \to \infty$ behavior of the eigenvector-eigenvalue equation for H.