COMP3711: Design and Analysis of Algorithms

Tutorial 3

HKUST

Question 1

Give asymptotic tight bounds for $T(n)$ by master theorem.

(a)

$$\begin{aligned} T(1) &= 1 \\ T(n) &= 3T(n/4) + n \qquad \text{if } n > 1 \end{aligned}$$

(b)

$$\begin{aligned} T(1) &= 1 \\ T(n) &= 3T(n/4) + 1 \qquad \text{if } n > 1 \end{aligned}$$

(c)

$$\begin{aligned} T(1) &= 1 \\ T(n) &= 4T(n/2) + n^2 \qquad \text{if } n > 1 \end{aligned}$$

(d)

$$\begin{aligned} T(1) &= 1 \\ T(n) &= 4T(n/3) + n^2 \qquad \text{if } n > 1 \end{aligned}$$

Solution 1

(a) This is in case 3 because $1 > \log_4 3$, so $T(n) = \Theta(n)$.
(b) This is in case 1 because $0 < \log_4 3$, so $T(n) = \Theta(n^{\log_4 3})$.
(c) This is in case 2 because $2 = \log_2 4$, so $T(n) = \Theta(n^2 \log n)$.
(d) This is in case 3 because $2 > \log_3 4$, so $T(n) = \Theta(n^2)$.

Question 2

Consider the HIRE-ASSISTANT algorithm in the lecture note, assuming that the candidates are presented in a random order.

(a) What is the probability that you hire exactly one time?

(b) What is the probability that you hire exactly n times?

Solution 2

(a) Probability that you hire exactly one time means that the first candidate you hire is the best one. So the probability is $\frac{1}{n}$.

(b) Probability that you hire exactly n times means that the candidates are in increasing order of their quality. So the probability is $\frac{1}{n!}$.

Question 3

Use indicator random variables to solve the following problem, which is known as the **hat-check problem**. Each of n customers gives a hat to a hat-check person at a restaurant. The hat-check person gives the hats back to the customers in a random order. What is the expected number of customers who get back their own hat?

Solution 3

Let $X_i = 1$ if the i-th customer get back his own hat, otherwise $X_i = 0$.
$E(X_i) = Pr(X_i = 1) = \frac{1}{n}$.

$$\begin{aligned} E(X) &= E\left(\sum_{i=1}^{n} X_i\right) \\ &= \sum_{i=1}^{n} E(X_i) \\ &= \sum_{i=1}^{n} \frac{1}{n} \\ &= 1 \end{aligned}$$

Note that the X_i's are not independent. For example, when $X_1 = X_2 = ... = X_{n-1} = 1$, then X_n must be 1 as well.

Let $A[1..n]$ be an array of n distinct numbers. If $i < j$ and $A[i] > A[j]$, then the pair (i, j) is called an **inversion** of A. Suppose that the elements of A form a uniform random permutation of $\langle 1, 2, ..., n \rangle$. Use indicator random variables to compute the expected number of inversions.

Solution 4

Let $X_{ij} = 1$ if $A[i] > A[j]$ where $i < j$, otherwise $X_{ij} = 0$.
$E(X_{ij}) = Pr(X_{ij} = 1) = \frac{1}{2}$.

$$
\begin{aligned}
E(X) &= E\left(\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}\right) \\
&= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E(X_{ij}) \\
&= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{2} \\
&= \frac{1}{2} \cdot \frac{n(n-1)}{2} \\
&= \frac{n(n-1)}{4}
\end{aligned}
$$