COMP3711: Design and Analysis of Algorithms

Tutorial 1

Asymptotic notations

Asymptotic upper bound

Definition (big-Oh)

$f(n) = O(g(n))$: There exists constant $c > 0$ and n_0 such that $f(n) \leq c \cdot g(n)$ for $n \geq n_0$.
Equivalent definition: $\lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$

Asymptotic lower bound

Definition (big-Omega)

$f(n) = \Omega(g(n))$: There exists constant $c > 0$ and n_0 such that $f(n) \geq c \cdot g(n)$ for $n \geq n_0$.
Equivalent definition: $\lim_{n\to\infty} \frac{f(n)}{g(n)} > 0$

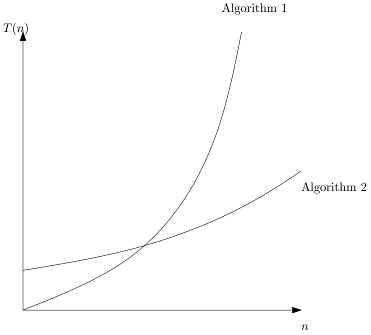
Asymptotic tight bound

Definition (big-Theta)

$f(n) = \Theta(g(n))$: $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

Comparing time complexity

Example:



Algorithm 2 is clearly superior

- $T(n)$ for Algorithm 1 is $O(n^3)$
- $T(n)$ for Algorithm 2 is $O(n^2)$
- Since n^3 grows much more rapidly, we expect Algorithm 1 to take much more time than Algorithm 2 when n increases

Some Basic mathematic background on exponentials

For all real $a \neq 0$, m and n, we have the following identities:

$$
\begin{aligned}
a^0 &= 1 \\
a^1 &= a \\
a^{-1} &= 1/a \\
(a^m)^n &= (a^n)^m = a^{mn} \\
a^m a^n &= a^{m+n} \\
a^{1/n} &= \sqrt[n]{a}
\end{aligned}
$$

Some Basic mathematic background on logarithms

For all real $a > 0, b > 0, c > 0$, and n:

$$
\begin{aligned}
a &= b^{\log_b a} \\
\log_c(ab) &= \log_c a + \log_c b \\
\log_b a^n &= n \log_b a \\
\log_b a &= \frac{\log_c a}{\log_c b} \\
\log_b(1/a) &= -\log_b a \\
\log_b a &= \frac{1}{\log_a b} \\
a^{\log_b n} &= n^{\log_b a}
\end{aligned}
$$

Question 1

For each of the following statement, answer whether the statement is true or false.

(a) $1000n + n \log n = O(n \log n)$.

(b) $n^2 + n \log(n^3) = O(n \log(n^3))$.

(c) $n^3 = \Omega(n)$.

(d) $n^2 + n = \Omega(n^3)$.

(e) $n^3 = O(n^{10})$.

(f) $n^3 + 1000n^{2.9} = \Theta(n^3)$

(g) $n^3 - n^2 = \Theta(n)$

Question 2

For each pair of expressions (A, B) below, indicate whether A is O, Ω, or Θ of B. Note that zero, one, or more of these relations may hold for a given pair; list all correct ones. Justify your answers.

(a) $A = n^3 + n \log n$; $\quad B = n^3 + n^2 \log n$.

(b) $A = \log \sqrt{n}$; $\quad B = \sqrt{\log n}$.

(c) $A = n \log_3 n$; $\quad B = n \log_4 n$.

(d) $A = 2^n$; $\quad B = 2^{n/2}$.

(e) $A = \log(2^n)$; $\quad B = \log(3^n)$.

Question 3

Suppose $T_1(n) = O(f(n))$ and $T_2(n) = O(f(n))$. Which of the following are true? Justify your answers.

(a) $T_1(n) + T_2(n) = O(f(n))$

(b) $\frac{T_1(n)}{T_2(n)} = O(1)$

(c) $T_1(n) = O(T_2(n))$

Question 4

Let $f(n)$ and $g(n)$ be non-negative functions. Using the basic definition of Θ-notation, prove that $\max(f(n), g(n)) = \Theta(f(n) + g(n))$.