COMP 3711: Mathematical Background

Common Log Identities:

$$\begin{aligned}
\log(a \cdot b) &= \log a + \log b \\
\log(a^b) &= b \log a \\
a^{\log_a b} &= b \\
a^{\log_b c} &= c^{\log_b a} \\
\log_a n &= \frac{\log_b n}{\log_b a} = \Theta(\log n) \\
\log(n!) &= \Theta(n \log n)
\end{aligned}$$

Common Summations: Let $c \neq 1$ be any positive constant and assume $n \geq 0$. The following are the most common summations that arise when analyzing algorithms and data structures. You should memorize their asymptotic values.

Name of Series	Formula	Closed-Form Solution	Asymptotic Form
Constant	$\sum_{i=1}^{n} 1$	$= n$	$\Theta(n)$
Arithmetic	$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n$	$= \frac{n(n+1)}{2}$	$\Theta(n^2)$
Polynomial	$\sum_{i=1}^{n} i^c = 1^c + 2^c + \cdots + n^c$	(none for general c)	$\Theta(n^{c+1})$
Geometric	$\sum_{i=0}^{n-1} c^i = 1 + c + c^2 + \cdots + c^{n-1}$	$= \frac{c^n - 1}{c - 1}$	$\Theta(c^n)$ $(c > 1)$ $\Theta(1)$ $(c < 1)$
Harmonic	$\sum_{i=1}^{n} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$	$= \ln n + O(1)$	$\Theta(\log n)$

Summations with general bounds: If summation bounds do not start at 1 and n, you can compute differences. For $1 \leq a \leq b$

$$\sum_{i=a}^{b} f(i) = \sum_{i=0}^{b} f(i) - \sum_{i=0}^{a-1} f(i).$$

(Simplified) Master Theorem for Recurrences: This is very useful when dealing with recurrences arising from divide-and-conquer algorithms. Let $a \geq 1$, $b > 1, c \geq 0$ be constants and let $T(n)$ be the recurrence $T(n) = aT(n/b) + n^c$, defined for $n \geq 1$.

Case 1: $c < \log_b a$ then $T(n)$ is $\Theta(n^{\log_b a})$.

Case 2: $c = \log_b a$ then $T(n)$ is $\Theta(n^c \log n)$.

Case 3: $c > \log_b a$ then $T(n)$ is $\Theta(n^c)$.